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Indecomposability and duality of tournaments

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Abstract

Let $T=(V,A)$ be a tournament. A subset X of V is an interval of T provided that for $a,b\in X$ and for $x\in V-X$, $(a,x)\in A$ if and only if $(b,x)\in A$. For example, $\emptyset$, $\{x\}$, where $x\in V$, and V are intervals of T, called trivial intervals. A tournament is said to be indecomposable if all of its intervals are trivial. In another respect, with each tournament $T=(V,A)$ is associated the dual tournament $T^\star=(V,A^\star)$ defined as: for $x,y\in V$, $(x,y)\in A^\star$ if $(y,x)\in A$. A tournament T is said to be self-dual if T and $T^\star$ are isomorphic. The paper characterizes the finite tournaments $T=(V,A)$ fulfilling: for every proper subset X of V, if the subtournament $T(X)$ of T is indecomposable, then $T(X)$ is self-dual. The corollary obtained is: given a finite and indecomposable tournament $T=(V,A)$, if T is not self-dual, then there is a subset X of V such that $6\leqslant|X|\leqslant 10$ and such that $T(X)$ is indecomposable without being self-dual. An analogous examination is made in the case of infinite tournaments. The paper concludes with an introduction of a new mode of reconstruction of tournaments from their proper and indecomposable subtournaments. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

A *tournament* T consists of a finite or infinite set V of *vertices* with a prescribed collection A of ordered pairs of distinct vertices, called the set of *arcs* of T, which satisfies: for $x,y\in V$, with $x\neq y$, $(x,y)\in A$ if and only if $(y,x)\notin A$. Such a tournament T is denoted by (V,A). A tournament $T=(V,A)$ is a *total order* provided that for $x,y,z\in V$, if $(x,y),(y,z)\in A$, then $(x,z)\in A$. For example, for each nonnegative integer

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n, the usual total order defined on $\{0,\dots,n-1\}$ is denoted by $O_n=(\{0,\dots,n-1\},A_n)$ where A_n is the following set of arcs: for $i,j\in\{0,\dots,n-1\}$, $(i,j)\in A_n$ if $i<j$. For convenience, given a tournament $T=(V,A)$, for $x,y\in V$, with $x\neq y$, $x\rightarrow y$ is used in the place of $(x,y)\in A$. Moreover, for $X\subseteq V$ and for $Y\subseteq V$, $X\rightarrow Y$ signifies that for every $x\in X$ and for every $y\in Y$, $x\rightarrow y$. If $X=\{x\}$ (resp. $Y=\{y\}$), then $X\rightarrow Y$ is denoted by $x\rightarrow Y$ (resp. $X\rightarrow y$). In another vein, if x is a vertex of a tournament $T=(V,A)$, then the set of $y\in V$ such that $x\rightarrow y$ (resp. $y\rightarrow x$) is denoted by $N_T^+(x)$ (resp. $N_T^-(x)$). Furthermore, if T is finite, then the *degree* of a vertex x of T, denoted by $d_T(x)$, is the cardinality of $N_T^+(x)$ and the *score* of T is the sequence $(d_T(x_1),\dots,d_T(x_n))$, denoted by $s(T)$, where $V=\{x_1,\dots,x_n\}$ and $d_T(x_1)\leqslant\cdots\leqslant d_T(x_n)$. The notions of isomorphism, of subtournament and of embedding are defined in the following manner. Firstly, let $T=(V,A)$ and $T'=(V',A')$ be two tournaments. A one-to-one correspondence f from V onto V' is an *isomorphism* from T onto T' provided that for $x,y\in V$, $(x,y)\in A$ if and only if $(f(x),f(y))\in A'$. The tournaments T and T' are then said to be *isomorphic*, which is denoted by $T\simeq T'$, if there is an isomorphism from T onto T'. Moreover, an isomorphism from a tournament T onto itself is called an *automorphism* of T and the set of the automorphisms of T is denoted by $\mathrm{Aut}(T)$. Secondly, given a tournament $T=(V,A)$, with each subset X of V is associated the subtournament $T(X)=(X,A\cap(X\times X))$ of T induced by X. For convenience, if $X\subseteq V$ (resp. $x\in V$), then the subtournament $T(V-X)$, where $V-X=\{x\in V: x\notin X\}$, (resp. $T(V-\{x\})$) is denoted by $T-X$ (resp. $T-x$). Once again, let T and T' be two tournaments. It is said that T *embeds* T' if T' is isomorphic to a subtournament of T. Finally, with each tournament $T=(V,A)$ is associated the *dual* tournament $T^\star=(V,A^\star)$ defined as: for $x,y\in V$, $(x,y)\in A^\star$ if $(y,x)\in A$. A tournament T is then said to be *self-dual* if T and $T^\star$ are isomorphic.

Self-duality and indecomposability play an important role in this paper. Given a tournament $T=(V,A)$, a subset X of V is an *interval* [6,12,20] (or an *autonomous subset* [7,13] or an *homogeneous subset* [9] or a *clan* [3] or a *module* [21] or a *partitive subset* [23]) of T provided that for every $x\in V-X$, either $x\rightarrow X$ or $X\rightarrow x$. This definition is a generalization of the classic notion of interval of a total order. Given a tournament $T=(V,A)$, $\emptyset$, V and $\{x\}$, where $x\in V$, are clearly intervals of T, called *trivial* intervals. A tournament is then said to be *indecomposable* [11,12,20] (or *prime* [2] or *primitive* [3] or *simple* [4,17]) if all of its intervals are trivial. Otherwise, a tournament $T=(V,A)$ which admits at least an interval X, such that $2\leqslant|X|<|V|$, is said to be *decomposable*. For example, if $n\geqslant3$, then $\{0,1\}$ is an interval of O_n and, consequently, O_n is decomposable. Lastly, the following relationship between indecomposability and self-duality must be emphasized. For any tournament T, T and $T^\star$ have the same intervals and, thus, T is indecomposable if and only if $T^\star$ is indecomposable.

In [19], Reid and Thomassen showed that if $T=(V,A)$ is a finite tournament, with $|V|\geqslant8$, such that for every $X\subset V$, where $X\subset V$ signifies that X is a proper subset of $V, T(X)$ is self-dual, then T is either a total order or T is obtained from a total order by reversing the arc between the minimum element and the maximum element. The aim of this paper is to examine, for each integer $n\geqslant5$, the class $\mathscr{S}_n$ of

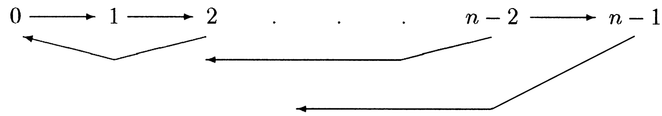

Fig. 1. P_n.

tournaments $T=(V,A)$ satisfying: $|V|=n$, T is indecomposable and for all $X\subset V$, if $T(X)$ is indecomposable, then $T(X)$ is self-dual. By using the characterization of the class $\overline{\mathscr{S}_n}$ of self-dual elements of $\mathscr{S}_n$, a similar examination is effected in the infinite case. The paper concludes with the introduction of some possible applications in the reconstruction scope.

2. Preliminaries

2.1. The strong connectivity

For each nonnegative integer n, P_n is the tournament defined on $\{0,\dots,n-1\}$ in the following way (see Fig. 1): given $i,j\in\{0,\dots,n-1\}$, with $i\neq j$, $i\to j$ if either $j=i+1$ or $j\leqslant i-2$.

The class of tournaments P_n, where $n\geqslant 5$, is denoted by $\mathscr{P}$ and the tournament P_3 is called *3-cycle*. Furthermore, given a set $S=\{x_0,\dots,x_{n-1}\}, P_{(x_0,\dots,x_{n-1})}$ denotes the only tournament defined on S such that the function which associates x_i with i, for $i\in\{0,\dots,n-1\}$, is an isomorphism from P_n onto $P_{(x_0,\dots,x_{n-1})}$.

The tournaments P_n allow for the definition of the relation $\mathscr{R}$ of *strong connectivity* as follows. Given vertices x and y of a tournament $T=(V,A)$, $x\mathscr{R}y$ if there are $x_0,\dots,x_k\in V$ and $y_0,\dots,y_l\in V$ fulfilling: $x_0=y_l=x$, $y_0=x_k=y$, $T(\{x_0,\dots,x_k\})=P_{(x_0,\dots,x_k)}$ and $T(\{y_0,\dots,y_l\})=P_{(y_0,\dots,y_l)}$. The relation $\mathscr{R}$ is an equivalence relation, the equivalence classes of which are called *strongly connected components* of T. The tournament T is then said to be *strongly connected* if it admits a single strongly connected component. To continue, the characterization of scores of strongly connected and finite tournaments is required.

Proposition 1. *Given integers* $0\leqslant d_1\leqslant\cdots\leqslant d_n$, $(d_1,\dots,d_n)$ *is the score of a strongly connected tournament if and only if*

$$d_1+\cdots+d_n=\binom{n}{2}$$

and for $i\in\{1,\dots,n-1\}$,

$$d_1+\cdots+d_i>\binom{i}{2}.$$

2.2. The indecomposable tournaments

The subsection commences with a direct consequence of the definitions of strong connectivity and indecomposability.

Lemma 1. *If $T=(V,A)$ is an indecomposable tournament such that $|V|\geqslant 3$, then T is strongly connected.*

The next corollary follows immediately from Proposition 1, from this lemma and from the definition of self-duality.

Corollary 1. *If $T=(V,A)$ is a finite and indecomposable tournament such that $|V|\geqslant 3$, then its score $s(T)=(d_1,\ldots,d_n)$ satisfies*

$$d_1+\cdots+d_n=\binom{n}{2}$$

and for $i\in\{1,\ldots,n-1\}$,

$$d_1+\cdots+d_i>\binom{i}{2}.$$

Moreover, if T is self-dual, then for $i\in\{1,\ldots,n\}$, $d_i+d_{n-i+1}=n-1$.

The following definitions are needed to construct indecomposable subtournaments of an indecomposable tournament.

Definition 1. Given a tournament $T=(V,A)$, with each subset X of V, such that $|X|\geqslant 3$ and $T(X)$ is indecomposable, are associated the following subsets of $V-X$:

- *Ext*(X) is the set of $x\in V-X$ such that $T(X\cup\{x\})$ is indecomposable.
- For every $u\in X$, $X(u)$ is the set of $x\in V-X$ such that $\{u,x\}$ is an interval of $T(X\cup\{x\})$.
- $[X]$ is the set of $x\in V-X$ such that X is an interval of $T(X\cup\{x\})$, that is to say, such that $x\rightarrow X$ or $X\rightarrow x$.

To continue, the properties of the intervals of a tournament, analogous to those of the intervals of a total order, are reviewed.

Proposition 2. *Let $T=(V,A)$ be a tournament. If X and Y are intervals of T, then $X\cap Y$ is an interval of T. Moreover, if $X\cap Y\neq\emptyset$ (resp. $X-Y\neq\emptyset$), then $X\cup Y$ (resp. $Y-X$) is an interval of T. Lastly, for each subset W of V, if X is an interval of T, then $X\cap W$ is an interval of $T(W)$.*

These properties allow for the establishment of the next lemma.

Lemma 2 (Ehrenfeucht and Rozenberg [3]). *Let $T=(V,A)$ be a tournament and let X be a subset of V such that $|X|\geqslant 3$ and $T(X)$ is indecomposable.*

1. *The family $\{X(u);\ u\in X\}\cup\{\mathrm{Ext}(X),[X]\}$ constitutes a partition of $V-X$.*
2. *Given $u\in X$, for all $x\in X(u)$ and for all $y\in V-(X\cup X(u))$, if $T(X\cup\{x,y\})$ is decomposable, then $\{u,x\}$ is an interval of $T(X\cup\{x,y\})$.*
3. *For every $x\in[X]$ and for every $y\in V-(X\cup[X])$, if $T(X\cup\{x,y\})$ is decomposable, then $X\cup\{y\}$ is an interval of $T(X\cup\{x,y\})$.*
4. *Given $x,\ y\in\mathrm{Ext}(X)$, with $x\neq y$, if $T(X\cup\{x,y\})$ is decomposable, then $\{x,y\}$ is an interval of $T(X\cup\{x,y\})$.*

The result below follows from Lemma 2, when it is assumed, in addition, that the considered tournament is indecomposable.

Corollary 2 (Ehrenfeucht and Rozenberg [3]). *Let $T=(V,A)$ be an indecomposable tournament. If X is a subset of V, such that $|X|\geqslant 3, |V-X|\geqslant 2$ and $T(X)$ is indecomposable, then there are distinct $x,y\in V-X$ such that $T(X\cup\{x,y\})$ is indecomposable.*

Because every indecomposable tournament embeds at least one 3-cycle, by utilizing Corollary 2 several times, the next corollary is obtained in the finite case.

Corollary 3 (Ehrenfeucht and Rozenberg [3]). *If $T=(V,A)$ is a finite and indecomposable tournament such that $|V|\geqslant 5$, then there are $x,y\in V$ such that $T-\{x,y\}$ is indecomposable.*

In the above corollary, we may have $x=y$. Schmerl and Trotter [20] improved this result as follows.

Proposition 3 (Schmerl and Trotter [20]). *If $T=(V,A)$ is a finite and indecomposable tournament such that $|V|\geqslant 7$, then there are distinct $x,\ y\in V$ such that $T-\{x,y\}$ is indecomposable.*

The subsection is completed by the below result which may be established by induction using Lemma 2.

Lemma 3. *If $n=3$ or if $n\geqslant 5$, then $P_n\in\overline{\mathscr{S}_n}$.*

2.3. The critical tournaments

A finite and indecomposable tournament $T=(V,A)$ is said to be *critical* if for all $x\in V$, $T-x$ is decomposable. In order to present the characterization of the critical

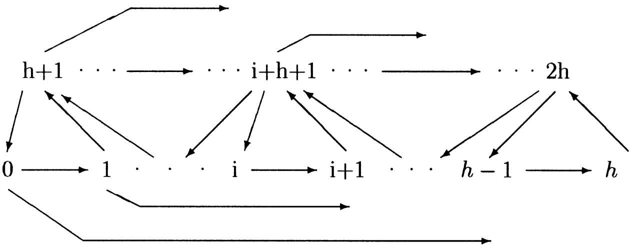

Fig. 2. T_{2h+1}.

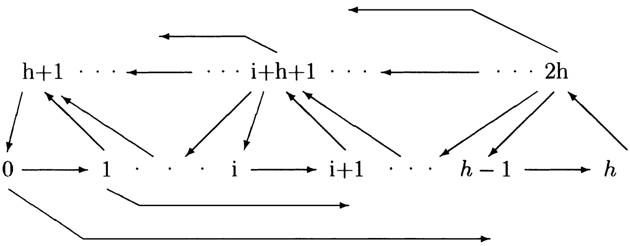

Fig. 3. U_{2h+1}.

tournaments attributed to Schmerl and Trotter [20], the following three classes of tournaments are introduced.

(i) $\mathscr{T}$ is the class of tournaments T_{2h+1}, where $h\geqslant 2$, defined on $\{0,\dots,2h\}$ as follows: given distinct $i,j\in\{0,\dots,2h\}$, $i\to j$ if $1\leqslant j-i\leqslant h$ where $j-i$ is considered modulo $(2h+1)$. In order to compare the tournaments T_{2h+1} with the tournaments of the next class, the following definition (see Fig. 2) is required: $T_{2h+1}(\{0,\dots,h\})=0<\cdots<h$, $T_{2h+1}(\{h+1,\dots,2h\})=h+1<\cdots<2h$ and for $i\in\{0,\dots,h-1\}$, $\{i+1,\dots,h\}\to i+h+1\to\{0,\dots,i\}$.

(ii) $\mathscr{U}$ is the class of tournaments U_{2h+1}, where $h\geqslant 2$, defined on $\{0,\dots,2h\}$ in the following manner (see Fig. 3): $U_{2h+1}(\{0,\dots,h\})=0<\cdots<h, (U_{2h+1})^{\star}(\{h+1,\dots,2h\})=h+1<\cdots<2h$ and for $i\in\{0,\dots,h-1\}, \{i+1,\dots,h\}\to i+h+1\to\{0,\dots,i\}$.

(iii) $\mathscr{V}$ is the class of tournaments V_{2h+1}, where $h\geqslant 2$, defined on $\{0,\dots,2h\}$ as follows (see Fig. 4): $V_{2h+1}(\{0,\dots,2h-1\})=0<1<\cdots<2h-1$ and $\{1,3,\dots,2h-1\}\to 2h\to\{0,2,\dots,2h-2\}$.

Theorem 1 (Schmerl and Trotter [20]). *Up to isomorphism, the only critical tournaments of cardinality* $\geqslant 5$ *are the tournaments* T_{2h+1}, U_{2h+1} *and* V_{2h+1} *where* $h\geqslant 2$.

For convenience, the class of critical tournaments of cardinality $n\geqslant 5$ is denoted by $\mathscr{C}_n$. In what follows, a new characterization of the critical tournaments, useful in the course of the examination of $\mathscr{S}_n$, is brought into play.

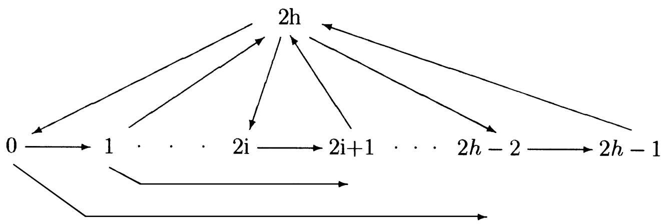

Fig. 4. V_{2h+1}.

Proposition 4. *Given a finite and indecomposable tournament T of cardinality $\geqslant 5, T$ is critical if and only if T does not embed indecomposable tournaments of cardinality* 6.

Proof. The indecomposable tournaments of cardinality 5 are critical because the four tournaments of cardinality 4 are decomposable. Consequently, only indecomposable tournaments $T=(V,A)$ of cardinality $\geqslant 6$ may be considered. If T is not critical, then either $|V|$ is even or $|V|$ is odd and, by the definition of critical tournaments, there is $x \in V$ such that $T-x$ is indecomposable. In both instances, there is $X \subseteq V$ such that $|X|$ is even, $|X| \geqslant 6$ and $T(X)$ is indecomposable. Then, by utilizing Proposition 3 several times, a subset Y of V is obtained such that $|Y|=6$ and $T(Y)$ is indecomposable.

Inversely, assume that there is $X \subseteq V$ such that $|X|=6$ and $T(X)$ is indecomposable. If $|V|$ is odd, then, by using Corollary 2 from X several times, a vertex x of T is obtained such that $T-x$ is indecomposable. If $|V|$ is even, then the conclusion is reached without utilizing the fact that T embeds an indecomposable tournament of cardinality 6. Indeed, since T is indecomposable, T embeds a 3-cycle from which, by using Corollary 2 several times, a subset Y of V may be constructed such that $|V-Y|=1$ and $T(Y)$ is indecomposable. □

The next three lemmas complete the examination of the critical tournaments. The last two will be used in Section 4.

Lemma 4. *For all integer $h \geqslant 2$, $\mathscr{C}_{2h+1} \subseteq \overline{\mathscr{S}_{2h+1}}$.*

Lemma 5. *Two critical tournaments of cardinality $\geqslant 5$ which do not belong to the same class $\mathscr{T}$, $\mathscr{U}$ or $\mathscr{V}$ are incomparable with respect to the embedding.*

Lemma 6. *Given an integer $n \geqslant 6$, the tournament P_n is incomparable with respect to the embedding to a critical tournament of cardinality $\geqslant 7$.*

3. Characterization of $\mathscr{S}_n$

The purpose of the section is to establish the next theorem, which constitutes the main result of this paper concerning the finite tournaments.

Theorem 2. *If $n \geqslant 11$, then, up to isomorphism, $\mathscr{S}_n = \mathscr{C}_n \cup \{P_n\}$.*

3.1. Examination of $\overline{\mathscr{S}_k}$ for $k = 5, \ldots, 10$

Since $V_5 \simeq P_5$, up to isomorphism, the elements of $\overline{\mathscr{S}_5}$ are P_5, T_5 and U_5. In order not to lengthen this subsection with technically simple verifications, the below characterizations of $\overline{\mathscr{S}_6}$ and of $\overline{\mathscr{S}_7}$ are demonstrated in the Appendix A.

(1) Up to isomorphism, $\overline{\mathscr{S}_6}$ contains three tournaments S_1^6, S_2^6 and S_3^6 defined on $\{0, \ldots, 5\}$ in the following way:
- $S_1^6(\{0, \ldots, 4\}) = P_5$ and, in S_1^6, $\{1, 2, 3\} \to 5 \to \{0, 4\}$.
- $S_2^6 = P_6$.
- $S_3^6(\{0, \ldots, 4\}) = U_5$ and, in S_3^6, $\{0, 2, 3\} \to 5 \to \{1, 4\}$.

(2) Up to isomorphism, $\overline{\mathscr{S}_7} - \mathscr{C}_7$ contains four tournaments S_1^7, S_2^7, S_3^7 and S_4^7 defined on $\{0, \ldots, 6\}$ in the following manner:
- $S_1^7 = P_7$.
- $S_2^7(\{0, \ldots, 5\}) = P_6$ and, in S_2^7, $\{3, 4, 5\} \to 6 \to \{0, 1, 2\}$.
- $S_3^7(\{0, \ldots, 5\}) = P_6$ and, in S_3^7, $\{0, 3, 4\} \to 6 \to \{1, 2, 5\}$.
- $S_4^7(\{0, \ldots, 5\}) = S_3^6$ and, in S_4^7, $\{1, 3, 5\} \to 6 \to \{0, 2, 4\}$.

To characterize the classes $\mathscr{S}_n$, three lemmas are used, preceded by one remark.

Remark 1. Given a finite and self-dual tournament $T = (V, A)$, let f be an isomorphism from T onto $T^\star$ and let Ω be an orbit of f such that $|\Omega| \geqslant 2$. Consider an element x of Ω and suppose, for example, that $x \to f(x)$. Since f is an isomorphism from T onto $T^\star$, for all $n \geqslant 0$, $f^{2n}(x) \to f^{2n+1}(x)$ and $f^{2n+2}(x) \to f^{2n+1}(x)$. As $f^{|\Omega|}(x) = x \to f^{|\Omega|+1}(x) = f(x)$, $|\Omega|$ is even. It follows that for every orbit Ω of f, either $|\Omega| = 1$ or $|\Omega|$ is even. Therefore, if $|V|$ is even, then for all $x \in V$, $f(x) \neq x$ and if $|V|$ is odd, then there is one and only one vertex x of T such that $f(x) = x$.

Lemma 7. *Given a tournament $T = (\{0, \ldots, n+1\}, A)$, where $n \geqslant 4$, if $T(\{0, \ldots, n\}) = P_{n+1}$ and if $T \simeq P_{n+2}$, then either $T = P_{n+2}$ or $T = P_{(n+1, 0, \ldots, n)}$.*

Lemma 8. *Given an element $T = (V, A)$ of $\overline{\mathscr{S}_{2n}}$, where $n \geqslant 3$, for every $x \in V$, if $T - x$ is indecomposable, then there are distinct y, $z \in V - \{x\}$ such that $d_{T-x}(y)$ is even and $d_{T-x}(z)$ is odd.*

Proof. Assume, on the contrary, that there is $x \in V$ such that $T - x$ is indecomposable and such that, for example, for all $y \in V - \{x\}$, $d_{T-x}(y)$ is even. Since $T \in \overline{\mathscr{S}_{2n}}$, there is an isomorphism f from T onto $T^\star$ and since $|V|$ is even, it follows from Remark 1 that $f(x) \neq x$. The set $V - \{x, f(x)\}$ may be thus decomposed in the following two manners:

(a) $V - \{x, f(x)\} = N^-_{T-x}[f(x)] \cup N^+_{T-x}[f(x)]$. As for every $y \in V - \{x\}$, $d_{T-x}(y)$ is even, for each $y \in N^-_{T-x}[f(x)]$ (resp. $y \in N^+_{T-x}[f(x)]$), $d_{T-\{x, f(x)\}}(y)$ is odd (resp. even).

(b) $V - \{x, f(x)\} = N^{-}_{T-f(x)}(x) \cup N^{+}_{T-f(x)}(x)$. Since $f_{/V-\{x\}}$ is an isomorphism from $T-x$ onto $T^{\star} - f(x)$, $T^{\star} - f(x)$ and, thus, $T - f(x)$ are indecomposable. Moreover, as $T \in \mathscr{S}_{2n}$, $T - f(x) \simeq [T - f(x)]^{\star} = T^{\star} - f(x)$. It follows that $T - f(x) \simeq T - x$ and, consequently, for all $y \in V - \{f(x)\}$, $d_{T-f(x)}(y)$ is even. As previously obtained, for all $y \in N^{-}_{T-f(x)}(x)$ (resp. $y \in N^{+}_{T-f(x)}(x)$), $d_{T-\{x,f(x)\}}(y)$ is odd (resp. even).

It ensues that $N^{-}_{T-x}[f(x)] = N^{-}_{T-f(x)}(x)$ and $N^{+}_{T-x}[f(x)] = N^{+}_{T-f(x)}(x)$, that is to say, $\{x, f(x)\}$ is an interval of T, which contradicts the indecomposability of T. $\square$

Lemma 9. *Given an integer $n \geqslant 5$, denote $\{0,\dots,n-1\}$ by X and consider a tournament $T = (\{0,\dots,n+1\}, A) \in \overline{\mathscr{S}_{n+2}}$. If $T - (n+1) = P_{n+1}$, then one and only one of the next assertions is satisfied.*

- $n+1 \in \mathrm{Ext}(X)$.
- $n+1 \in [X]$ *and* $T = P_{n+2}$.
- $n+2 \in X(2)$, $n = 6$ *and* $T = D_8$ *where* D_8 *is the tournament defined on* $\{0,\dots,7\}$ *as follows*: $D_8(\{0,\dots,6\}) = P_7$ *and* $\{1,4,5\} \to 7 \to \{0,2,3,6\}$.
- $n+1 \in X(4)$, $n = 5$ *and* $T = S^7_2$.
- $n+1 \in X(1)$, $n = 5$ *and* $T = S^7_3$.

Proof. It follows from Lemma 2 applied to $T(X) = P_n$ that if $n+1 \notin \mathrm{Ext}(X)$, then either there is $u \in X$ such that $n+1 \in X(u)$ or $n+1 \in [X]$. However, if $n+1 \in X(0) \cup X(n-2)$, then either $n \to n+1$ and T is decomposable or $n+1 \to n$ and T is not self-dual. Furthemore, if $n+1 \in X(i)$, where $i \in \{2,\dots,n-5\} \cup \{n-3\}$, then either $n \to n+1$ and T is decomposable or $n+1 \to n$ and $T - i$ is indecomposable without being self-dual. Therefore, it suffices to envisage the below cases.

- $n+1 \in X(1)$. Since T is indecomposable and since $n \to 1$, $n+1 \to n$. If $1 \to n+1$ or if $n+1 \to 1$, with $n \geqslant 6$, then T is not self-dual. It follows that $n+1 \to 1$, $n = 5$ and, thus, $T = S^7_3$.
- $n \geqslant 6$ *and* $n+1 \in X(n-4)$. As T is indecomposable and as $n \to n-4$, $n+1 \to n$. If $n \geqslant 7$ or if $n = 6$, with $2 \to 7$, then T is not self-dual. It ensues that $n = 6$, $7 \to 2$ and, hence, $T = D_8$.
- $n+1 \in X(n-1)$. Since T is indecomposable and since $n-1 \to n$, $n \to n+1$. If $n+1 \to n-1$ or if $n-1 \to n+1$, with $n \geqslant 6$, then T is not self-dual. It follows that $n = 5$, $n-1 \to n+1$ and, thus, $T = S^7_2$.
- $n+1 \in [X]$. If $X \to n+1$, then either $n \to n+1$ and T is decomposable or $n+1 \to n$ and T is not self-dual. It ensues that $n+1 \to X$. As T is indecomposable, $n \to n+1$ and, hence, $T = P_{n+2}$. $\square$

It is now possible to characterize the elements of the classes $\overline{\mathscr{S}_8}$, $\overline{\mathscr{S}_9}$ and $\overline{\mathscr{S}_{10}}$.

Proposition 5. *Up to isomorphism, $\overline{\mathscr{S}_8}$ contains two tournaments S_1^8 and S_2^8 defined on $\{0,\dots,7\}$ as follows*:

- $S_1^8 = P_8$.
- $S_2^8(\{0,\dots,6\}) = P_7$ *and* $\{0,3,4,6\} \to 7 \to \{1,2,5\}$.

Before demonstrating this proposition, the following remark is made.

Remark 2. Given $T = (\{0,\dots,7\}, A) \in \overline{\mathscr{S}_8}$, for every $x \in \{0,\dots,7\}$, if $T - x$ is indecomposable, then $T - x \in \overline{\mathscr{S}_7} = \{S_i^7; 1 \leqslant i \leqslant 4\} \cup \{T_7, U_7, V_7\}$ and it follows from Lemma 8 that $T - x \in \{S_i^7; 1 \leqslant i \leqslant 3\}$. In another vein, since T is of even cardinality, T is not critical and there is $x \in \{0,\dots,7\}$ such that $T - x$ is indecomposable. Assume that $x = 7$ and that $T - 7 = P_7, S_2^7$ or S_3^7. In the three instances, $T(X) = P_6$ where $X = \{0,\dots,5\}$. If $7 \in \mathrm{Ext}(X)$, that is to say, if $T - 6$ is indecomposable, then, as previously seen, $T - 6 \simeq P_7, S_2^7$ or S_3^7. If $T - 6 \simeq P_7$, then it follows from Lemma 7 that $T - 6 = P_{(0,\dots,5,7)}$ or $P_{(7,0,\dots,5)}$. If $T - 6 \simeq S_2^7$ (resp. S_3^7), then let f be an isomorphism from T onto S_2^7 (resp. S_3^7). As for $x \in \{0,\dots,5\}$, $S_2^7 - x$ (resp. $S_3^7 - x$) is decomposable, $f(7) = 6$. Therefore, $f_{/\{0,\dots,5\}} \in \mathrm{Aut}(P_6)$ and, thus, $f_{/\{0,\dots,5\}} = \mathrm{Id}$. It ensues that $\{3,4,5\} \to 7 \to \{0,1,2\}$ (resp. $\{0,3,4\} \to 7 \to \{1,2,5\}$).

Proof of Proposition 5. It ensues from Lemma 3 that $P_8 \in \overline{\mathscr{S}_8}$ and it may be directly verified that $S_2^8 \in \overline{\mathscr{S}_8}$. Inversely, given $T = (\{0,\dots,7\}, A) \in \overline{\mathscr{S}_8}$, as in the above remark, the following three cases are distinguished:

1. There is $x \in \{0,\dots,7\}$ such that $T - x \simeq P_7$. Suppose that $x = 7$ and that $T - 7 = P_7$. It follows from Lemma 9 that either $T = P_8$ or D_8 or $7 \in \mathrm{Ext}(X)$ where $X = \{0,\dots,5\}$. If $T = D_8$, then the permutation $(0,1,2)(3,7)(4,5,6)$ is an isomorphism from T onto S_2^8. If $7 \in \mathrm{Ext}(X)$, then, by Remark 2, it suffices to consider four subcases.
   - If $T - 6 = P_{(0,\dots,5,7)}$, then $\{6,7\}$ is an interval of T, which contradicts the indecomposability of T.
   - If $T - 6 = P_{(7,0,\dots,5)}$ and if $7 \to 6$, then $T - 5$ is indecomposable without being self-dual. It follows that if $T - 6 = P_{(7,0,\dots,5)}$, then $6 \to 7$ and, thus, $T \simeq P_8$.
   - If $\{3,4,5\} \to 7 \to \{0,1,2\}$, then T is not self-dual.
   - If $\{0,3,4\} \to 7 \to \{1,2,5\}$ and if $7 \to 6$, then T is not self-dual. It ensues that if $\{0,3,4\} \to 7 \to \{1,2,5\}$, then $6 \to 7$ and, hence, $T = S_2^8$.
2. There is $x \in \{0,\dots,7\}$ such that $T - x \simeq S_2^7$. Assume that $x = 7$ and that $T - 7 = S_2^7$. If $7 \in \bigcup_{i \in \{0,\dots,5\}} X(i)$, where $X = \{0,\dots,5\}$, then, by interchanging T and $T^\star$, it may be supposed that $7 \in X(0) \cup X(1) \cup X(2)$. Similarly, if $7 \in [X]$, then, by considering $T^\star$ in the place of T, it may be assumed that $7 \to X$. Yet, if $7 \in X(0) \cup X(2)$ (resp. $7 \in [X]$), then either $6 \to 7$ (resp. $7 \to 6$) and T is decomposable or $7 \to 6$ (resp. $6 \to 7$) and T is not self-dual. Moreover, if $7 \in X(1)$, then either $6 \to 7$ and T is decomposable or $7 \to 6$ and $T - 1$ is indecomposable without being self-dual. Consequently, $7 \in \mathrm{Ext}(X)$, that is to say, $T - 6$ is indecomposable. By Remark 2, either $T - 6 \simeq P_7$ or $\{3,4,5\} \to 7 \to \{0,1,2\}$ or $\{0,3,4\} \to 7 \to \{1,2,5\}$. The

first instance allows for a return to the previous case. In the second one, $\{6,7\}$ is an interval of T, which contradicts the indecomposability of T. In the third one, whatever the arc between 6 and 7, T is not self-dual.

3. There is $x \in \{0,\dots,7\}$ such that $T - x \simeq S_3^7$. Assume that $x = 7$ and that $T - 7 = S_3^7$. If $7 \in \bigcup_{i \in \{0,\dots,5\}} X(i)$, where $X = \{0,\dots,5\}$, then, by interchanging T and $T^\star$, it may be assumed that $7 \in X(0) \cup X(1) \cup X(2)$. In the same way, if $7 \in [X]$, then, by considering $T^\star$ in the place of T, it may be supposed that $7 \to X$. However, if $7 \in X(0) \cup [X]$, then either $7 \to 6$ and T is decomposable or $6 \to 7$ and T is not self-dual. Furthermore, if $7 \in X(i)$, where $i \in \{1,2\}$, then either $6 \to 7$ and T is decomposable or $7 \to 6$ and $T - i$ is indecomposable without being self-dual. In consequence, $7 \in \mathrm{Ext}(X)$, that is to say, $T - 6$ is indecomposable. By Remark 2, either $T - 6 \simeq P_7$ (resp. $T - 6 \simeq S_2^7$), which allows for a return to the first (resp. second) case or $\{0,3,4\} \to 7 \to \{1,2,5\}$ and $\{6,7\}$ is an interval of T, which contradicts the indecomposability of T. $\square$

Proposition 6. *Up to isomorphism, $\overline{\mathscr{S}_9}$ contains, on the one hand, the three critical tournaments T_9, U_9, V_9 and, on the other hand, the tournaments S_1^9 and S_2^9 defined on $\{0,\dots,8\}$ as follows*:

- $S_1^9 = P_9$.
- $S_2^9(\{0,\dots,7\}) = S_2^8$ *and* $\{1,4,5,7\} \to 8 \to \{0,2,3,6\}$.

Before demonstrating this proposition, a remark is made.

Remark 3. Given $T = (\{0,\dots,8\}, A) \in \overline{\mathscr{S}_9}$, if T is not critical, then there is $x \in \{0,\dots,8\}$ such that $T - x$ is indecomposable and, thus, such that $T - x \in \overline{\mathscr{S}_8}$. By Proposition 5, it is supposed that $x = 8$ and that $T - 8 = P_8$ or S_2^8. In both instances, $T(X) = P_7$ is indecomposable where $X = \{0,\dots,6\}$. If $8 \in \mathrm{Ext}(X)$, that is to say, if $T - 7$ is indecomposable, then, as previously seen, $T - 7 \simeq P_8$ or S_2^8. If $T - 7 \simeq P_8$, then it follows from Lemma 7 that $T - 7 = P_{(0,\dots,6,8)}$ or $P_{(8,0,\dots,6)}$. If $T - 7 \simeq S_2^8$, then let f be an isomorphism from $T - 7$ onto S_2^8. Since $\{x \in \{0,\dots,7\}\colon S_2^8 - x \simeq P_7\} = \{3,7\}$, either $f(8) = 3$ or $f(8) = 7$. In the first instance, as $S_2^8 - 3 = P_{(1,2,0,7,5,6,4)}$, $f_{/\{0,1,2\}\cup\{4,5,6\}} = (0,1,2)(4,5,6)$, $f(3) = 7$ and, consequently, $\{1,4,5\} \to 8 \to \{0,2,3,6\}$. In the second one, $S_2^8 - 7 = P_6$, $f_{/\{0,\dots,6\}} = \mathrm{Id}$ and, hence, $\{0,3,4,6\} \to 8 \to \{1,2,5\}$.

Proof of Proposition 6. It follows from Lemmas 3 and 4 that $\mathscr{C}_9 \cup \{P_9\} \subseteq \overline{\mathscr{S}_9}$ and it may be directly verified that $S_2^9 \in \overline{\mathscr{S}_9}$. Inversely, given $T = (\{0,\dots,8\}, A) \in \overline{\mathscr{S}_9}$, if T is not critical, then, as in the above remark, two cases are distinguished.

1. There is $x \in \{0,\dots,8\}$ such that $T - x \simeq P_8$. Assume that $x = 8$ and that $T - 8 = P_8$. It ensues from Lemma 9 that either $T = P_9$ or $8 \in \mathrm{Ext}(X)$ where $X = \{0,\dots,6\}$. By Remark 3, if $8 \in \mathrm{Ext}(X)$, then it suffices to consider four subcases.
   - If $T - 7 = P_{(0,\dots,6,8)}$, then $\{7,8\}$ is an interval of T, which contradicts the indecomposability of T.

- If $T-7=P_{(8,0,\ldots,6)}$ and if $8\to 7$, then $T-6$ is indecomposable without being self-dual. It follows that if $T-7=P_{(8,0,\ldots,6)}$, then $7\to 8$ and, thus, $T\simeq P_9$.
- If $\{0,3,4,6\}\to 8\to\{1,2,5\}$, then T is not self-dual.
- If $\{1,4,5\}\to 8\to\{0,2,3,6\}$, then either $8\to 7$ and T is not self-dual or $7\to 8$ and $T-2$ is indecomposable without being self-dual.

2. There is $x\in\{0,\ldots,8\}$ such that $T-x\simeq S_2^8$. Suppose that $x=8$ and that $T-8=S_2^8$. By considering the next five subcases, it will be established that $8\notin(\bigcup_{i\in X}X(i))\cup[X]$ where $X=\{0,\ldots,6\}$.
   - If $8\in X(0)$, then either $8\to 7$ and T is decomposable or $7\to 8$ and T is not self-dual.
   - If $8\in X(i)$, where $i\in\{1,2\}$ (resp. $i\in\{4,6\}$), then either $7\to 8$ (resp. $8\to 7$) and T is decomposable or $8\to 7$ (resp. $7\to 8$) and $T-i$ is indecomposable without being self-dual.
   - If $8\in X(3)$, then either $8\to 7$ and T is decomposable or $7\to 8$ and $T-\{3,6\}$ is indecomposable without being self-dual.
   - If $8\in X(5)$, then either $7\to 8$ and T is decomposable or $8\to 7$ and $T-0$ is indecomposable without being self-dual.
   - If $8\to X$ (resp. $X\to 8$), then either $8\to 7$ (resp. $7\to 8$) and T is decomposable or $7\to 8$ (resp. $8\to 7$) and T is not self-dual.

   By Lemma 2, $8\in\mathrm{Ext}(X)$, that is to say, $T-7$ is indecomposable. It then follows from Remark 3 that either $T-7\simeq P_8$ or $\{0,3,4,6\}\to 8\to\{1,2,5\}$ or $\{1,4,5\}\to 8\to\{0,2,3,6\}$. The first instance allows for a return to the first case. In the second instance, $\{7,8\}$ is an interval of T, which contradicts the indecomposability of T. In the third one, if $8\to 7$, then $T-5$ is indecomposable without being self-dual. It follows that if $\{1,4,5\}\to 8\to\{0,2,3,6\}$, then $7\to 8$ and, hence, $T=S_2^9$. □

Proposition 7. *Up to isomorphism,* $\overline{\mathscr{S}_{10}}=\{P_{10}\}$.

Proof. It follows from Lemma 3 that $P_{10}\in\overline{\mathscr{S}_{10}}$. Inversely, let $T=(\{0,\ldots,9\},A)$ be an element of $\overline{\mathscr{S}_{10}}$. By Proposition 3, there are distinct x, $y\in\{0,\ldots,9\}$ such that $T-\{x,y\}$ is indecomposable. As $T\in\overline{\mathscr{S}_{10}}$, $T-\{x,y\}\in\overline{\mathscr{S}_8}$ and, by Proposition 5, $T-\{x,y\}$ is isomorphic to $S_1^8=P_8$ or to S_2^8. In both instances, there is $Y\subseteq\{0,\ldots,9\}-\{x,y\}$ such that $T(Y)\simeq P_7$. It then ensues from Corollary 2 that there are distinct u, $v\in\{0,\ldots,9\}-Y$ such that $T(Y\cup\{u,v\})$ is indecomposable and, therefore, $T(Y\cup\{u,v\})\in\overline{\mathscr{S}_9}$. By Proposition 4, since $T(Y\cup\{u,v\})$ embeds $T(Y)\simeq P_7$ and, hence, P_6, $T(Y\cup\{u,v\})$ is not critical. Therefore, $T(Y\cup\{u,v\})$ is isomorphic to $S_1^9=P_9$ or to S_2^9. As $s(S_2^9)=(2,2,2,4,4,4,6,6,6)$, it follows from Lemma 8 that $T(Y\cup\{u,v\})\not\simeq S_2^9$. Consequently, it may be supposed that $Y\cup\{u,v\}=\{0,\ldots,8\}$ and that $T(\{0,\ldots,8\})=P_9$. By denoting $\{0,\ldots,7\}$ by X, it is obtained, by Lemma 9, that $T\simeq P_{10}$ or that $9\in\mathrm{Ext}(X)$. So, assume that $9\in\mathrm{Ext}(X)$ that is to say that $T-8$ is indecomposable. As $T-8$ embeds $T(\{0,\ldots,5\})=P_6$, it follows from Proposition 4 that $T-8$ is not critical. As before, it ensues that $T-8\simeq P_9$ and, by Lemma 7, either $T-8=P_{(0,\ldots,7,9)}$ or $T-8=P_{(9,0,\ldots,7)}$. In the first instance, $\{8,9\}$ is an interval of T and, thus, T would be decomposable.

In the second one, it may be shown that if $9 \to 8$, then $T-7$ is indecomposable without being self-dual. Indeed, if f is an isomorphism from $T-7$ onto $T^\star-7$, then f switches $\{i \in \{0,\dots,9\}-\{7\}: d_{T-7}(i) \in \{1,2\}\} = \{0,1,9\}$ and $\{i \in \{0,\dots,9\}-\{7\}: d_{T-7}(i) \in \{6,7\}\} = \{5,6,8\}$; however, $T(\{0,1,9\}) \simeq P_3$ and $T(\{5,6,8\}) \simeq O_3$. In another respect, by denoting $\{0,\dots,6\}$ by Z, $T(Z)=P_7$ is indecomposable and as $T(Z\cup\{9\}) = P_{(9,0,\dots,6)}$, $9 \in \mathrm{Ext}(Z)$. Finally, since $9 \to 8 \to Z$, it follows from Lemma 2 that $T(Z\cup\{8,9\}) = T-7$ is indecomposable. Therefore, $T-8 = P_{(9,0,\dots,7)}$, $8 \to 9$ and, as $T-9=P_9$, $T \simeq P_{10}$. $\square$

The Propositions 5–7 allow for the demonstration of Theorem 2 in the following manner.

Proof of Theorem 2. It ensues from Lemmas 3 and 4 that for all $n \geqslant 5$, $\mathscr{C}_n \cup \{P_n\} \subseteq \overline{\mathscr{S}_n}$. Inversely, by using the former proposition, it is sufficient to prove that for every $n \geqslant 10$, if $\overline{\mathscr{S}_n} - \mathscr{C}_n = \{P_n\}$, then $\mathscr{S}_{n+1} - \mathscr{C}_{n+1} = \{P_{n+1}\}$. So, let $T = (\{0,\dots,n\}, A)$ be an element of $\mathscr{S}_{n+1} - \mathscr{C}_{n+1}$ where $n \geqslant 10$. Since T is not critical, there is $i \in \{0,\dots,n\}$ such that $T-i$ is indecomposable and, therefore, $T-i \in \overline{\mathscr{S}_n}$. In consequence, if $n+1$ is odd, then $\mathscr{C}_n = \emptyset$ and $T-i \simeq P_n$. On the contrary, if $n+1$ is even, then, by using Proposition 3 from T several times, a subset X of $\{0,\dots,n\}$ is obtained such that $|X|=8$ and $T(X)$ is indecomposable. Consequently, $T(X) \in \overline{\mathscr{S}_8}$ and it follows from Proposition 5 that $T(X) \simeq S_1^8$ or S_2^8. As $S_1^8 - 7 = S_2^8 - 7 = P_7$, there is $x \in X$ such that $T(X-\{x\}) \simeq P_7$ and, by utilizing Corollary 2 from $X-\{x\}$ several times, a subset Y of $\{0,\dots,n\}$ is obtained such that $|Y|=n$, $X-\{x\} \subseteq Y$ and $T(Y)$ is indecomposable. In consequence, $T(Y) \in \overline{\mathscr{S}_n}$ and, since $X-\{x\} \subseteq Y$, $T(Y)$ embeds P_7 and, thus, P_6. It then follows from Proposition 4 that $T(Y)$ is not critical and, therefore, $T(Y) \simeq P_n$. It ensues that, whatever the parity of n, there is $i \in \{0,\dots,n\}$ such that $T-i \simeq P_n$. Suppose, for example, that $i=n$ and that $T-n=P_n$. To commence, it may be proven that $n \notin X(0)$ where $X=\{0,\dots,n-2\}$. Indeed, if $n \in X(0)$, then $T(\{0,\dots,n-2\}\cup\{n\}) = P_{(n,1,\dots,n-2)}$ and as T is indecomposable, $n \to n-1$. Since $T(\{1,\dots,n-3\}\cup\{n\}) = P_{(n,1,\dots,n-3)}$, by denoting $\{1,\dots,n-3\}$ by Y, $T(Y)$ is indecomposable and $n \in \mathrm{Ext}(Y)$. Moreover, as $T-n=P_n$, $n-1 \to Y$. It then follows from Lemma 2 that, as $n \to n-1$, $T(Y\cup\{n-1,n\}) = T-\{0,n-2\}$ is indecomposable. In another vein, if f is an isomorphism from $T-\{0,n-2\}$ onto $T^\star - \{0,n-2\}$, then, since $\{i \in \{1,\dots,n\}-\{n-2\}: d_{T-\{0,n-2\}}(i) \in \{1,2\}\} = \{1,2,n\}$ and since $\{i \in \{1,\dots,n\}-\{n-2\}: d_{T-\{0,n-2\}}(i) \in \{n-4,n-3\}\} = \{n-4,n-3,n-1\}$, $f(\{1,2,n\}) = \{n-4,n-3,n-1\}$; yet, $T(\{1,2,n\}) \simeq P_3$ and $T(\{n-4,n-3,n-1\}) \simeq O_3$. In consequence, $T-\{0,n-2\}$ would be indecomposable without being self-dual and, therefore, $n \notin X(0)$. At present, it may be shown that $n \notin \bigcup_{u\in\{1,\dots,n-2\}} X(u)$ and that if $n \in [X]$, then $n \to X$. Indeed, if $n \in X(u)$, where $u \in \{1,\dots,n-2\}$, (resp. $n \in [X]$), then, by denoting $\{1,\dots,n-2\}$ by Y, $n \in Y(u)$ (resp. $n \in [Y]$). Furthermore, if $\{u,n\}$ (resp. $Y\cup\{n-1\}$) is an interval of $T-0$, then, as $n \in X(u)$ (resp. $n \in [X]$), $\{u,n\}$ (resp. $X\cup\{n-1\}$) is an interval of T, which contradicts the indecomposability of T. Consequently, $\{u,n\}$ (resp. $Y\cup\{n-1\}$) is not an interval of $T-0$ and it ensues from Lemma 2 that $T-0$ is indecomposable and, hence, $T-0 \in \overline{\mathscr{S}_n}$. Let T' be the unique tournament such that g is an isomorphism from $T-0$ onto T' where g is the function which associates $i-1$ with each $i\in\{1,\dots,n\}$. As $n\in Y(u)$ (resp. $n\in[Y]$), by denoting $\{0,\dots,n-3\}$ by X', in T', $n-1\in X'(u-1)$ (resp. $n-1\in[X']$). Moreover, since $T-n=P_n$, $T'-(n-1)=P_{n-1}$. As $n\geqslant 10$ and as $n-1\notin \mathrm{Ext}(X')$, it follows from Lemma 9 applied to T' and to X' that $T'=P_n$. In consequence, in T', $n-1\to X'$ and, thus, in T, $n\to Y$ and $n\to X$. Since T is indecomposable, $n-1\to n$ and, therefore, $T=P_{n+1}$. Finally, assume that $n\in\mathrm{Ext}(X)$ that is to say that $T-(n-1)\in\overline{\mathscr{S}_n}$. It is then proven that $T\simeq P_{n+1}$. As $T-(n-1)$ embeds $T(X)=P_{n-1}$, $T-(n-1)$ embeds P_6 and, by Proposition 4, $T-(n-1)$ is not critical. Consequently, $T-(n-1)\simeq P_n$ and it follows from Lemma 7 that either $T-(n-1)=P_{(0,\dots,n-2,n)}$ or $T-(n-1)=P_{(n,0,\dots,n-2)}$. Since, in the first instance, $\{n-1,n\}$ is an interval of T, $T-(n-1)=P_{(n,0,\dots,n-2)}$. In order to conclude, it will be established that if $n\to n-1$, then $T-(n-2)$ is indecomposable without being self-dual. Indeed, as $T-n=P_n$, $T(Y)=P_{n-4}$ is indecomposable, where $Y=\{0,\dots,n-3\}$, and $n-1\to Y$. Furthermore, since $T-(n-1)=P_{(n,0,\dots,n-2)}$, $T(Y\cup\{n\})=P_{(n,0,\dots,n-3)}$ is indecomposable or, equivalently, $n\in\mathrm{Ext}(Y)$. As $n\to n-1\to Y$, it ensues from Lemma 2 that $T(Y\cup\{n-1,n\})=T-(n-2)$ is indecomposable. However, $T-(n-2)$ is not self-dual as, if f is an isomorphism from $T-(n-2)$ onto $T^\star-(n-2)$, then f switches $\{i\in\{0,\dots,n\}-\{n-2\}\colon d_{T-(n-2)}(i)\in\{1,2\}\}=\{0,1,n\}$ and $\{i\in\{0,\dots,n\}-\{n-2\}\colon d_{T-(n-2)}(i)\in\{n-3,n-2\}\}=\{n-4,n-3,n-1\}$; yet, $T(\{0,1,n\})\simeq P_3$ and $T(\{n-4,n-3,n-1\})\simeq O_3$. If follows that $n-1\to n$ and, hence, that $T\simeq P_{n+1}$. $\square$

The corollary below of Theorem 2 concludes the section.

Corollary 4. *Given a finite and indecomposable tournament $T=(V,A)$, if T is not self-dual, then there is a subset X of V such that $6\leqslant|X|\leqslant 10$ and such that $T(X)$ is indecomposable without being self-dual.*

Proof. Proceed by induction on $|V|$. Clearly, the statement is satisfied if $6\leqslant|V|\leqslant 10$. At present, assume that $|V|\geqslant 11$. By Theorem 2, all of the elements of $\mathscr{S}_{|V|}$ are self-dual. Consequently, $T\notin\mathscr{S}_{|V|}$, that means, there is a proper subset W of V such that $T(W)$ is indecomposable without being self-dual. By induction hypothesis, there exists a subset X of W such that $6\leqslant|X|\leqslant 10$ and such that $T(X)$ is indecomposable without being self-dual. $\square$

4. An analogous examination in the infinite case

Let $\mathscr{S}_\infty$ be the class of infinite and indecomposable tournaments $T=(V,A)$ fulfilling: for every proper subset X of V, if $T(X)$ is indecomposable, then $T(X)$ is self-dual. The following theorem links the class $\mathscr{S}_\infty$ to the classes $\mathscr{S}_n$ examined in the preceding section.

Theorem 3 (Ille [11]). *Given an infinite tournament $T=(V,A)$, T is indecomposable if and only if for every finite subset F of V, there is a finite subset G of V such that $F\subseteq G$ and $T(G)$ is indecomposable.*

It ensues that the class $\mathscr{S}_\infty$ is included in the class $\mathscr{S}_\infty^F$ of infinite tournaments $T=(V,A)$ fulfilling: for all finite subsets F of V, there is a finite subset G of V such that $|G|\geqslant 5$, $F\subseteq G$ and $T(G)\in\overline{\mathscr{S}_{|G|}}$. For convenience, the following notations will be used.

- Given an infinite tournament $T=(V,A)$, the family of its finite and indecomposable subtournaments of cardinality $\geqslant 5$ is denoted by $\mathscr{F}(T)$.
- For $\mathscr{X}=\mathscr{P},\mathscr{T},\mathscr{U}$ or $\mathscr{V}$, $\mathscr{X}_\infty$ (resp. $\mathscr{X}_\infty^F$) is the class of tournaments T of $\mathscr{S}_\infty$ (resp. $\mathscr{S}_\infty^F$) such that $\mathscr{F}(T)\subseteq\mathscr{X}$.

The following result follows from Theorems 2 and 3, and from Lemmas 5 and 6.

Corollary 5. *The family $\{\mathscr{P}_\infty,\mathscr{T}_\infty,\mathscr{U}_\infty,\mathscr{V}_\infty\}$ (resp. $\{\mathscr{P}_\infty^F,\mathscr{T}_\infty^F,\mathscr{U}_\infty^F,\mathscr{V}_\infty^F\}$) constitutes a partition of $\mathscr{S}_\infty$ (resp. $\mathscr{S}_\infty^F$).*

The purpose of the section is to characterize the classes $\mathscr{P}_\infty^F,\mathscr{T}_\infty^F,\mathscr{U}_\infty^F$ and $\mathscr{V}_\infty^F$. In order to examine the elements of $\mathscr{P}_\infty^F$, these tournaments are used.

- The usual order on the nonnegative integers (resp. on the integers) is denoted by ω (resp. $\omega^\star+\omega$). For convenience, ω will also designate the set of nonnegative integers.
- The tournament P_ω (resp. $P_{\omega^\star+\omega}$) is defined on the nonnegative integers (resp. on the integers) as follows: given nonnegative integers (resp. integers) i, j, with $i\neq j$, $i\to j$ if $j=i+1$ or $j\leqslant i-2$.

Proposition 8. *Up to isomorphism, $\mathscr{P}_\infty^F=\{P_\omega,(P_\omega)^\star,P_{\omega^\star+\omega}\}$.*

Proof. It will be verified, for example, that $P_\omega\in\mathscr{P}_\infty^F$. Given a finite set F of nonnegative integers such that $|F|\geqslant 5$, by denoting $\{\min(F),\dots,\max(F)\}$ by G, $P_\omega(G)\simeq P_{\max(F)-\min(F)+1}$ and, therefore, $P_\omega(G)\in\mathscr{P}$. Inversely, let $T=(V,A)$ be an element of $\mathscr{P}_\infty^F$. In the first place, the following is noticed. Given a finite subset X of V such that $T(X)$ is indecomposable, as $T\in\mathscr{P}_\infty^F$, $T(X)\simeq P_{|X|}$ and, thus, $T(X)=P_{(x_0,\dots,x_n)}$ where $X=\{x_0,\dots,x_n\}$. It will then be shown that $\{X^-,X^+,\mathrm{Ext}^-(X),\mathrm{Ext}^+(X)\}$ constitutes a partition of $V-X$ where $X^-=\{x\in V-X\colon x\to X\}$, $X^+=\{x\in V-X\colon X\to x\}$, $\mathrm{Ext}^-(X)=\{x\in V-X\colon T(X\cup\{x\})=P_{(x,x_0,\dots,x_n)}\}$ and $\mathrm{Ext}^+(X)=\{x\in V-X\colon T(X\cup\{x\})=P_{(x_0,\dots,x_n,x)}\}$. Indeed, for all $x\in V-X$, as $T\in\mathscr{P}_\infty^F$, there is a finite subset $Y=\{y_0,\dots,y_m\}$ of V such that $X\cup\{x\}\subseteq Y$ and $T(Y)=P_{(y_0,\dots,y_m)}$. If $\min(\{i\in\{0,\dots,m\}\colon y_i\in X\})$ (resp. $\max(\{i\in\{0,\dots,m\}\colon y_i\in X\})$) is denoted by α (resp. β), then, since $T(X)$ is strongly connected, $X=\{y_\alpha,\dots,y_\beta\}$, $T(X)=P_{(y_\alpha,\dots,y_\beta)}$ and, necessarily, for $i\in\{\alpha,\dots,\beta\}$, $y_i=x_{\alpha-i}$. By denoting by γ the single element of $\{0,\dots,m\}-\{\alpha,\dots,\beta\}$ such that $x=y_\gamma$, either

$\gamma \in \{0,\dots,\alpha-2\}$ and $x \in X^+$ or $\gamma=\alpha-1$ and $x \in \mathrm{Ext}^-(X)$ or $\gamma=\beta+1$ and $x \in \mathrm{Ext}^+(X)$ or $\gamma \in \{\beta+2,\dots,m\}$ and $x \in X^-$. In consequence, $\{X^-, X^+, \mathrm{Ext}^-(X), \mathrm{Ext}^+(X)\}$ constitutes a partition of $V-X$. Furthermore, if x, $y \in \mathrm{Ext}^-(X)$, with $x \neq y$, then, by denoting $X \cup \{x\}$ by Y, $y \in Y(x)$ and, thus, $\{Y^-, Y^+, \mathrm{Ext}^-(Y), \mathrm{Ext}^+(Y)\}$ would not be a partition of $V-Y$. It follows that $|\mathrm{Ext}^-(X)| \leqslant 1$ and, in the same manner, $|\mathrm{Ext}^+(X)| \leqslant 1$. Also, note that if $x \in \mathrm{Ext}^-(X)$ and if $y \in \mathrm{Ext}^+(X)$, then $T(X \cup \{x,y\}) = P_{(x,x_0,\dots,x_n,y)}$. Indeed, by denoting $X \cup \{x\}$ by Y, as $Y^- \subseteq X^-$ and as $Y^+ \subseteq X^+$, $y \notin Y^- \cup Y^+$. Furthermore, if $y \in \mathrm{Ext}^-(Y)$, that is to say, if $T(X \cup \{x,y\}) = P_{(y,x,x_0,\dots,x_n)}$, then $y \in X^+$. Consequently, $y \in \mathrm{Ext}^+(Y)$ or, equivalently, $T(X \cup \{x,y\}) = P_{(x,x_0,\dots,x_n,y)}$. Finally, $|\mathrm{Ext}^-(X) \cup \mathrm{Ext}^+(X)| \leqslant 2$ and $T[\mathrm{Ext}^-(X) \cup X \cup \mathrm{Ext}^+(X)] \simeq P_{|\mathrm{Ext}^-(X) \cup X \cup \mathrm{Ext}^+(X)|}$. This leads to the definition by induction of the sequence $(X_n)_{n\geqslant 0}$ of finite subsets of V in the following way: X_0 is a finite subset of V, with $|X| \geqslant 5$, such that $T(X_0) \simeq P_{|X_0|}$ and for all $n \geqslant 0$, $X_{n+1} = \mathrm{Ext}^-(X_n) \cup X_n \cup \mathrm{Ext}^+(X_n)$. It follows from the definition of the sequence $(X_n)_{n\geqslant 0}$ that if $\mathrm{Ext}^-(X_p) = \emptyset$ (resp. $\mathrm{Ext}^+(X_p) = \emptyset$), where $p \geqslant 0$, then for every $n \geqslant p$, $\mathrm{Ext}^-(X_n) = \emptyset$ (resp. $\mathrm{Ext}^+(X_n) = \emptyset$). It suffices, therefore, to consider four cases.

- If for every $n \geqslant 0$, $\mathrm{Ext}^-(X_n) \neq \emptyset$ and $\mathrm{Ext}^+(X_n) \neq \emptyset$, then $T(\bigcup_{n\geqslant 0} X_n) \simeq P_{\omega^\star+\omega}$.
- If there is $p \geqslant 0$ such that $\mathrm{Ext}^-(X_p) = \emptyset$ and if for each $n \geqslant 0$, $\mathrm{Ext}^+(X_n) \neq \emptyset$, then $T(\bigcup_{n\geqslant 0} X_n) \simeq P_\omega$.
- If there is $p \geqslant 0$ such that $\mathrm{Ext}^+(X_p) = \emptyset$ and if for all $n \geqslant 0$, $\mathrm{Ext}^-(X_n) \neq \emptyset$, then $T(\bigcup_{n\geqslant 0} X_n) \simeq (P_\omega)^\star$.
- If there is $p \geqslant 0$ such that $\mathrm{Ext}^-(X_p) = \mathrm{Ext}^+(X_p) = \emptyset$, then $T(\bigcup_{n\geqslant 0} X_n) \simeq P_{|X_p|}$.

To conclude, it is proven that $\bigcup_{n\geqslant 0} X_n$ is an interval of T. Indeed, if $x \in V - (\bigcup_{n\geqslant 0} X_n)$, then, in particular, $x \notin X_1$ and, hence, $x \in (X_0)^- \cup (X_0)^+$. Since for every $n \geqslant 0$, $(X_{n+1})^- \subseteq (X_n)^-$ and $(X_{n+1})^+ \subseteq (X_n)^+$, either $x \in \bigcap_{n\geqslant 0} (X_n)^-$, that is to say, $x \to \bigcup_{n\geqslant 0} X_n$ or $x \in \bigcap_{n\geqslant 0} (X_n)^+$, that is to say, $\bigcup_{n\geqslant 0} X_n \to x$. Consequently, $\bigcup_{n\geqslant 0} X_n$ is an interval of T and as T is indecomposable, $V = \bigcup_{n\geqslant 0} X_n$. Finally, since V is infinite, it ensues from the four previously considered cases that T is isomorphic to P_ω, to $(P_\omega)^\star$ or to $P_{\omega^\star+\omega}$. $\square$

In order to characterize the elements of the classes $\mathcal{T}_\infty^F$ and $\mathcal{U}_\infty^F$, these definitions and lemmas are utilized. Given a total order $O = (V, A)$, a subset X of V is an *initial interval* (resp. *final interval*) of O provided that for every $x \in X$ and for every $y \in V$, if $y \to x$ (resp. $x \to y$), then $y \in X$. For example, $\emptyset$ and V are initial intervals of O, called *trivial* initial intervals. In another respect, both tournaments Δ^- and Δ^+ defined on $\{0,\dots,3\}$ as follows, are called *diamonds*: $\Delta^-(\{0,1,2\}) = \Delta^+(\{0,1,2\}) = P_3$ and, in Δ^- (resp. Δ^+), $3 \to \{0,1,2\}$ (resp. $\{0,1,2\} \to 3$). Furthermore, a vertex x of a tournament $T = (V, A)$ is said to be a *diamond's center* if there is a subset X of V containing x and an isomorphism f from $T(X)$ onto Δ^- or onto Δ^+ such that $f(x) = 3$. For convenience, the set of vertices of T which are diamond's centers is denoted by $\mathcal{C}(T)$. Finally, the characterization of indecomposable tournaments, which do not embed diamonds, is recalled.

Lemma 10 (Gnanvo and Ille [8]). *Given a finite and indecomposable tournament T of cardinality $\geqslant 5$, $T \in \mathscr{T}$ if and only if T does not embed diamonds.*

The following proposition generalizes the construction of the tournaments T_n represented in Fig. 2.

Proposition 9. *Given an infinite tournament $T=(V,A)$, $T \in \mathscr{T}_{\infty}^{F}$ if and only if there is a subset X of V such that $T(X)$ is a total order and there is a function φ which associates with each $x \in V-X$ a nontrivial initial interval $\varphi(x)$ of $T(X)$ and which satisfies the following assertions*:

1. *For all $x \in V-X$, $X-\varphi(x) \to x \to \varphi(x)$.*
2. *For every distinct x, $y \in V-X$, $x \to y$ if and only if $\varphi(x) \subset \varphi(y)$.*
3. *For all distinct a, $b \in X$, if $a \to b$, then there is $x \in V-X$ such that $a \in \varphi(x)$ and $b \notin \varphi(x)$.*

Proof. Let $T=(V,A)$ be an element of $\mathscr{T}_{\infty}^{F}$. By Zorn's lemma, there is a subset X of V fulfilling: $T(X)$ is a total order and for every subset Y of V, if $T(Y)$ is a total order and if $X \subseteq Y$, then $X=Y$. Moreover, as $T \in \mathscr{T}_{\infty}^{F}$, it follows from Theorem 3 and from Lemma 10 that T does not embed diamonds. Firstly, it may then be established that for all $x \in V-X$, $N_T^{+}(x) \cap X$ and $N_T^{-}(x) \cap X$ are intervals of $T(X)$. Indeed, if, for example, $N_T^{+}(x) \cap X$ is not an interval of $T(X)$, then there are $a,b,c \in X$ such that $a \to b \to c$ and $b \to x \to \{a,c\}$. In consequence, since $T(\{a,b,x\}) \simeq P_3$ and since $\{a,b,x\} \to c$, T would embed the diamond Δ^{+}. It ensues that $N_T^{+}(x) \cap X$ and $N_T^{-}(x) \cap X$ are intervals of $T(X)$ and, hence, that $N_T^{+}(x) \cap X$ is an initial or a final interval of $T(X)$. Yet, if $N_T^{+}(x) \cap X$ is a final interval of $T(X)$, then $T(X \cup \{x\})$ is a total order, which contradicts the maximality of X. Consequently, $N_T^{+}(x) \cap X$ is a nontrivial initial interval of $T(X)$ which is denoted by $\varphi(x)$. It follows that for every $x \in V-X$, $X-\varphi(x) \to x \to \varphi(x)$. Secondly, let x and y be elements of $V-X$ such that $\varphi(x) \subset \varphi(y)$. There is, thus, $a \in \varphi(x) \cap \varphi(y)$ and $b \in \varphi(y)-\varphi(x)$ such that $a \to b$. As T does not embed diamonds, as $T(\{a,b,x\}) \simeq P_3$ and as $y \to \{a,b\}$, $x \to y$. It ensues that for each nontrivial initial interval I of $T(X)$, $\varphi^{-1}(I)=\{x \in V-X\colon \varphi(x)=I\}$ is an interval of T. Indeed, if $a \in X$, then either $a \in I$ and $\varphi^{-1}(I) \to a$ or $a \notin I$ and $a \to \varphi^{-1}(I)$. Moreover, if $y \in V-(\varphi^{-1}(I) \cup X)$, then, since the initial intervals of a total order are comparable with respect to the inclusion, either $\varphi(y) \subset I$ and, by the foregoing, $y \to \varphi^{-1}(I)$ or $I \subset \varphi(y)$ and $\varphi^{-1}(I) \to y$. As T is indecomposable, $|\varphi^{-1}(I)| \leqslant 1$ or, equivalently, φ is injective. In consequence, for all distinct $x,y \in V-X$, either $\varphi(x) \subset \varphi(y)$ and $x \to y$ or $\varphi(y) \subset \varphi(x)$ and $y \to x$. It ensues that for every distinct $x,y \in V-X$, if $x \to y$, then $\varphi(x) \subset \varphi(y)$. Finally, let a and b be elements of X such that $a \to b$. As the tournament T is indecomposable, $[a,b]=\{a,b\} \cup \{c \in X\colon a \to c \to b\}$ is not an interval of T. However, if $x \in V-X$ such that $b \in \varphi(x)$ (resp. $a \notin \varphi(x)$), then $x \to [a,b]$ (resp. $[a,b] \to x$). Therefore, since $[a,b]$ is an interval of $T(X)$, there is $x \in V-X$ such that $a \in \varphi(x)$ and $b \notin \varphi(x)$.

Inversely, let $T=(V,A)$ be an infinite tournament such that there is a function φ which associates with each $x\in V-X$, where X is a subset of V such that $T(X)$ is a total order, a nontrivial initial interval of $T(X)$ and which fulfills the above three assertions. Given a finite subset F of V, such that $|F\cap X|\geqslant 3$ and $|F-X|\geqslant 2$, the elements of $F\cap X$ are denoted by $a_0,\dots,a_p$ where for $i\in\{0,\dots,p-1\}$, $a_i\to a_{i+1}$. By hypothesis, for $i\in\{0,\dots,p-1\}$, the set U_i of elements x of $V-X$, such that $a_{i+1}\to x\to a_i$, is not empty. For $i\in\{0,\dots,p-1\}$, the subset F_i of U_i is then defined as follows: $F_i=\emptyset$ if $F\cap U_i\neq\emptyset$ and $F_i=\{x_i\}$, where $x_i\in U_i$, if $F\cap U_i=\emptyset$. Now, consider the finite subset $G=F\cup(\bigcup_{i\in\{0,\dots,p-1\}}F_i)$ of V and denote the elements of $G-X$ by $x_0,\dots,x_q$ where for $i\in\{0,\dots,q-1\}$, $x_i\to x_{i+1}$. By hypothesis, for $i\in\{0,\dots,q-1\}$, the set X_i of elements a of X, such that $x_{i+1}\to a\to x_i$, is not empty. For $i\in\{0,\dots,q-1\}$, the subset G_i of X_i is then defined in the following way: $G_i=\emptyset$ if $G\cap X_i\neq\emptyset$ and $G_i=\{a_i\}$, where $a_i\in X_i$, if $G\cap X_i=\emptyset$. Finally, consider the finite subset $H=G\cup(\bigcup_{i\in\{0,\dots,q-1\}}G_i)$ of V and denote the elements of $H\cap X$ by $\alpha_0,\dots,\alpha_n$ where for $i\in\{0,\dots,n-1\}$, $\alpha_i\to\alpha_{i+1}$. Moreover, for $i\in\{0,\dots,n-1\}$, the set of elements x of $H-X$, such that $\alpha_{i+1}\to x\to\alpha_i$, is denoted by W_i. In order to demonstrate that $T(H)\simeq T_{2n+1}$, it is sufficient to verify that for $i\in\{0,\dots,n-1\}$, $|W_i|=1$. Indeed, given $i\in\{0,\dots,n-1\}$, if $|W_i|\geqslant 2$, then, as $H-X=G-X$, there is $j\in\{0,\dots,q-1\}$ such that $x_j,x_{j+1}\in W_i$. Yet, by the definition of H, there is $\alpha\in H\cap X$ such that $x_{j+1}\to\alpha\to x_j$. Since $\alpha_{i+1}\to x_{j+1}\to\alpha$, $\alpha\to\alpha_{i+1}$ and, since $\alpha\to x_j\to\alpha_i$, $\alpha_i\to\alpha$. In consequence, α_i and α_{i+1} would not be consecutive elements of $T(H\cap X)$. To conclude, it will be shown, by considering the next four cases, that for $i\in\{0,\dots,n-1\}$, $W_i\neq\emptyset$.

- If $\alpha_i,\alpha_{i+1}\in F$, then, by the definition of G, $G\cap W_i\neq\emptyset$.
- If $\alpha_i\in F$ and if $\alpha_{i+1}\notin F$, then, by the definition of H, there is $j\in\{0,\dots,q-1\}$ such that $\{\alpha_{i+1}\}=G_j$ and, hence, such that $x_{j+1}\to\alpha_{i+1}\to x_j$. As $\alpha_i\to\alpha_{i+1}$, $x_{j+1}\to\alpha_i$ and, by the definition of H, as $G_j\neq\emptyset$, that is to say, as $G\cap X_j=\emptyset$, $\alpha_i\notin X_j$ and, thus, $x_j\to\alpha_i$. Therefore, $\alpha_{i+1}\to x_j\to\alpha_i$ or, equivalently, $x_j\in W_i$.
- If $\alpha_i\notin F$ and if $\alpha_{i+1}\in F$, then, by the definition of H, there is $j\in\{0,\dots,q-1\}$ such that $\{\alpha_i\}=G_j$ and, thus, such that $x_{j+1}\to\alpha_i\to x_j$. As $\alpha_i\to\alpha_{i+1}$, $\alpha_{i+1}\to x_j$ and, by the definition of H, as $G_j\neq\emptyset$, that is to say, as $G\cap X_j=\emptyset$, $\alpha_{i+1}\notin X_j$ and, hence, $\alpha_{i+1}\to x_{j+1}$. Therefore, $\alpha_{i+1}\to x_{j+1}\to\alpha_i$ or, equivalently, $x_{j+1}\in W_i$.
- If $\alpha_i,\alpha_{i+1}\notin F$, then, by the definition of F, there is $j\in\{0,\dots,q-1\}$ (resp. $k\in\{0,\dots,q-1\}$) such that $\{\alpha_i\}=G_j$ (resp. $\{\alpha_{i+1}\}=G_k$) and, thus, such that $x_{j+1}\to\alpha_i\to x_j$ (resp. $x_{k+1}\to\alpha_{i+1}\to x_k$). As $\alpha_i\to\{x_j,\alpha_{i+1}\}$, $\alpha_{i+1}\to x_j$ and, as $x_{k+1}\to\alpha_{i+1}$, $x_j\to x_{k+1}$ or, equivalently, $j\leqslant k$. Furthermore, since $\alpha_i\neq\alpha_{i+1}$, $j\neq k$ and, hence, $j\leqslant k-1$. Finally, if $j\leqslant k-2$, then, by the definition of H, there would be $\alpha\in H\cap X$ such that $x_{j+2}\to\alpha\to x_{j+1}$. As $j+2\leqslant k$ and as $\alpha_{i+1}\to x_k$, $\alpha_{i+1}\to x_{j+2}$. It follows that $\alpha_{i+1}\to x_{j+2}\to\alpha$ and, thus, $\alpha\to\alpha_{i+1}$. Similarly, since $\alpha\to x_{j+1}\to\alpha_i$, $\alpha_i\to\alpha$ and, therefore, α_i and α_{i+1} would not be consecutive elements of $T(H\cap X)$. It ensues that $j=k-1$ and, hence, $\alpha_{i+1}\to x_k=x_{j+1}\to\alpha_i$ or, equivalently, $x_k\in W_i$. $\square$

In order to characterize the elements of $\mathscr{U}_\infty^F$, the following properties of the tournaments U_{2h+1} will be used.

Lemma 11. *Given an integer $h \geqslant 2$, if X is a subset of $\{0,\dots,2h\}$ such that $U_{2h+1}(X) \simeq P_3$, then $|X \cap \{0,\dots,h\}| = 2$ and $|X \cap \{h+1,\dots,2h\}| = 1$. Moreover, $\mathscr{C}(U_{2h+1}) = \{h+1,\dots,2h\}$ and for every subset Y of $\{0,\dots,2h\}$, such that $|Y| \geqslant 5$ and $U_{2h+1}(Y) \simeq U_{|Y|}$, $\mathscr{C}[U_{2h+1}(Y)] = \{h+1,\dots,2h\} \cap Y$.*

The next proposition generalizes the construction of the tournaments U_n represented in Fig. 3.

Proposition 10. *Given an infinite tournament $T = (V,A)$, $T \in \mathscr{U}_\infty^F$ if and only if there is a subset X of V such that $T(X)$ is a total order and there is a function φ which associates with each $x \in V - X$ a nontrivial initial interval $\varphi(x)$ of $T(X)$ and which fulfills the following assertions*:

1. *For all $x \in V - X$, $X - \varphi(x) \to x \to \varphi(x)$.*
2. *For every distinct $x, y \in V - X$, $x \to y$ if and only if $\varphi(y) \subset \varphi(x)$.*
3. *For all distinct $a, b \in X$, if $a \to b$, then there is $x \in V - X$ such that $a \in \varphi(x)$ and $b \notin \varphi(x)$.*

Proof. Given a tournament $T = (V,A) \in \mathscr{U}_\infty^F$, to commence, it will be established that for every finite subset F of V such that $|F| \geqslant 5$, if $T(F) \simeq U_{|F|}$, then $\mathscr{C}[T(F)] = \mathscr{C}(T) \cap F$. It is clear that $\mathscr{C}[T(F)] \subseteq \mathscr{C}(T) \cap F$. Inversely, for all $x \in \mathscr{C}(T) \cap F$, there is a subset Δ_x of V such that $x \in \Delta_x$ and $T(\Delta_x) \simeq \Delta^-$ or Δ^+. As $T \in \mathscr{U}_\infty^F$, there is a finite subset G of V such that $F \cup \Delta_x \subseteq G$ and $T(G) \simeq U_{|G|}$. In consequence, $x \in \mathscr{C}[T(G)]$ and since, by Lemma 11, $\mathscr{C}[T(F)] = \mathscr{C}[T(G)] \cap F$, $x \in \mathscr{C}[T(F)]$. At present, the tournament $T' = (V, A')$ is defined in the following manner: for every distinct $x, y \in V$ such that $(x,y) \in A$, $(x,y) \in A'$ if $\{x,y\} - \mathscr{C}(T) \neq \emptyset$ and $(y,x) \in A'$ if $\{x,y\} \subseteq \mathscr{C}(T)$. As $T \in \mathscr{U}_\infty^F$, for each finite subset F of V, there is a finite subset G of V such that $F \subseteq G$ and $T(G) \simeq U_{|G|}$. Since for all $h \geqslant 2$, T_{2h+1} is obtained from U_{2h+1} by reversing all of the arcs contained in $\{h+1,\dots,2h\} = \mathscr{C}(U_{2h+1})$ and since, by the foregoing, $\mathscr{C}[T(G)] = \mathscr{C}(T) \cap G$, $T'(G) \simeq T_{|G|}$ and, therefore, $T' \in \mathscr{T}_\infty^F$. Moreover, assume that there is a subset X of $V - \mathscr{C}(T)$ such that $T(X) \simeq P_3$. As $T \in \mathscr{U}_\infty^F$, there is a finite subset G of V such that $X \subseteq G$ and $T(G) \simeq U_{|G|}$. However, since, by that which precedes, $\mathscr{C}[T(G)] = \mathscr{C}(T) \cap G$, $X \subseteq G - \mathscr{C}[T(G)]$, which contradicts Lemma 11. It ensues that $T[V - \mathscr{C}(T)]$ and, thus, $T'[V - \mathscr{C}(T)]$ are total orders. Furthermore, as $T \in \mathscr{U}_\infty^F$, for every $x \in \mathscr{C}(T)$, there is a finite subset G of V such that $x \in G$ and $T(G) \simeq U_{|G|}$. Similarly, since $\mathscr{C}[T(G)] = \mathscr{C}(T) \cap G$, $x \in \mathscr{C}[T(G)]$. In addition, since for all $i \in \{0,\dots,h-1\}$, where $h \geqslant 2$, $U_{2h+1}(\{i, i+1, i+h+1\}) \simeq P_3$, there are distinct y, $z \in G - \mathscr{C}[T(G)]$ such that $T(\{x,y,z\}) \simeq P_3$. As $\mathscr{C}[T(G)] = \mathscr{C}(T) \cap G$, $y, z \in V - \mathscr{C}(T)$ and, hence, $T[(V - \mathscr{C}(T)) \cup \{x\}]$ embeds P_3. It follows that $T'[(V - \mathscr{C}(T)) \cup \{x\}]$ embeds P_3 and, consequently, $V - \mathscr{C}(T)$ fulfills: $T'[V - \mathscr{C}(T)]$ is a total order and for every subset Y of V, if $T'(Y)$ is a total order and if $V - \mathscr{C}(T) \subseteq Y$, then $Y = V - \mathscr{C}(T)$. As in the proof of Proposition 9, from the subset $X = V - \mathscr{C}(T)$ of V, a function φ may be constructed which associates with each $x \in V - X$ a nontrivial initial interval $\varphi(x) = N^+_{T'}(x) \cap X$ of $T'(X)$ and which satisfies the three assertions enunciated in Proposition 9. In particular, firstly, as for all $x \in V - X$, $T(X \cup \{x\}) = T'(X \cup \{x\})$, $\varphi(x)$ is a nontrivial initial interval of $T(X)$ such that, in T, $X - \varphi(x) \to x \to \varphi(x)$. Secondly, for every distinct $x, y \in V - X$, $x \to y$ in T' if and only if $\varphi(x) \subset \varphi(y)$. It follows from the definition of T' that for all distinct $x, y \in V - X$, $x \to y$ in T if and only if $\varphi(y) \subset \varphi(x)$. Finally, for every distinct $a, b \in X$, if $a \to b$ in T, that is to say, if $a \to b$ in T', then there is $x \in V - X$ such that $a \in \varphi(x)$ and $b \notin \varphi(y)$.

Inversely, let $T = (V, A)$ be an infinite tournament such that there is a function φ which associates with each $x \in V - X$, where X is a subset of V such that $T(X)$ is a total order, a nontrivial initial interval of $T(X)$ and which fulfills the three assertions of the above statement. As before, the tournament $T' = (V, A')$ is defined in the following way: for all distinct $x,\ y \in V$ such that $(x, y) \in A$, $(x, y) \in A'$ if $\{x, y\} \cap X \neq \emptyset$ and $(y, x) \in A'$ if $\{x, y\} \cap X = \emptyset$. It then ensues from Proposition 9 that $T' \in \mathscr{T}^F_\infty$. Indeed, in the course of the demonstration of Proposition 9, a finite subset H of V was constructed, from a finite subset F of V, such that $F \subseteq H$ and $T'(H) \simeq T_{|H|}$. More precisely, by denoting the elements of $H \cap X$ by $\alpha_0, \dots, \alpha_n$, where for $i \in \{0, \dots, n-1\}$, $\alpha_i \to \alpha_{i+1}$, it was established that for $i \in \{0, \dots, n-1\}$, the set of elements x of $H - X$, such that, in T', $\{\alpha_n, \dots, \alpha_{i+1}\} \to x \to \{\alpha_0, \dots, \alpha_i\}$, contains a single element denoted, for convenience, by α_{i+n+1}. In consequence, the function which associates α_i with each $i \in \{0, \dots, 2n\}$ is an isomorphism from T_{2n+1} onto $T'(H)$. Finally, as $T(H)$ is obtained from $T'(H)$ by reversing all of the arcs contained in $H - X = \{\alpha_{n+1}, \dots, \alpha_{2n}\}$, $T(H) \simeq U_{2n+1}$. □

The next proposition generalizes the construction of the tournaments V_n represented in Fig. 4.

Proposition 11. *Given an infinite tournament $T = (V, A)$, $T \in \mathscr{V}^F_\infty$ if and only if there is a vertex α of T such that $T - \alpha$ is a total order fulfilling the following assertions*:

1. *For every distinct $a,\ b \in V - \{\alpha\}$ such that $a \to b$, $[a, b]_{T-\alpha} = \{a, b\} \cup \{c \in V - \{\alpha\}: a \to c \to b\}$ is not an interval of T.*
2. *If $T - \alpha$ admits a minimum (resp. maximum) element denoted by* $\min(T - \alpha)$ *(resp.* $\max(T - \alpha)$*), then $\alpha \to \min(T - \alpha)$ (resp. $\max(T - \alpha) \to \alpha$).*

Proof. For convenience, for all tournaments $T' \in \mathscr{V}$, $\alpha(T')$ will denote the unique vertex of T' such that $T' - \alpha(T')$ is a total order. Now, let $T = (V, A)$ be an element of $\mathscr{V}^F_\infty$ and let $F \subseteq G$ be finite subsets of V such that $|F| \geqslant 5$, $T(F) \simeq V_{|F|}$ and $T(G) \simeq V_{|G|}$. As $T(F)$ is indecomposable and as $T(G) - \alpha[T(G)]$ is a total order, $\alpha[T(G)] \in F$. Consequently, $T(F) - \alpha[T(G)]$ is a total order and it follows from the uniqueness of $\alpha[T(F)]$ that $\alpha[T(F)] = \alpha[T(G)]$. As $T \in \mathscr{V}^F_\infty$, there is a finite subset F of V such that

$|F|\geqslant 5$ and $T(F)\simeq V_{|F|}$. Moreover, for every finite subset X of $V-\{\alpha[T(F)]\}$, there is a finite subset G of V such that $F\cup X\subseteq G$ and $T(G)\simeq V_{|G|}$. By the foregoing, since $\alpha[T(F)]=\alpha[T(G)]$, $X\subseteq G-\alpha[T(G)]$ and, thus, $T(X)$ is a total order. Therefore, by denoting $\alpha[T(F)]$ by α, we obtain that $T-\alpha$ is a total order and by utilizing the indecomposability of T, it is easy to verify that $T-\alpha$ satisfies the above two assertions.

Inversely, let $T=(V,A)$ be an infinite tournament which admits a vertex α such that $T-\alpha$ is a total order which fulfills both assertions of the above statement. Given an integer $k\geqslant 2$ and elements $a_0,\dots,a_k$ of $V-\{\alpha\}$ such that for $i\in\{0,\dots,k-1\}$, $a_i\rightarrow a_{i+1}$, the subsets $X_{\min},X_{\max},X_0,\dots,X_{k-1}$ of $V-\{\alpha\}$ are defined as follows.

- If $\alpha\rightarrow a_0$ (resp. $a_k\rightarrow\alpha$), then $X_{\min}=\emptyset$ (resp. $X_{\max}=\emptyset$).
- If $a_0\rightarrow\alpha$ (resp. $\alpha\rightarrow a_k$), then a_0 (resp. a_k) is not the minimum (resp. maximum) element of $T-\alpha$ or, equivalently, there is $b\in V-\{\alpha\}$ such that $b\rightarrow a_0$ (resp. $a_k\rightarrow b$). As $[b,a_0]_{T-\alpha}$ (resp. $[a_k,b]_{T-\alpha}$) is not an interval of T, there is $x_{\min}\in[b,a_0]_{T-\alpha}$ (resp. $x_{\max}\in[a_k,b]_{T-\alpha}$) such that $\alpha\rightarrow x_{\min}$ (resp. $x_{\max}\rightarrow\alpha$) and $X_{\min}=\{x_{\min}\}$ (resp. $X_{\max}=\{x_{\max}\}$).
- For $i\in\{0,\dots,k-1\}$, either $\{a_i,a_{i+1}\}$ is not an interval of $T(\{a_i,a_{i+1},\alpha\})$ and $X_i=\emptyset$ or $\{a_i,a_{i+1}\}$ is an interval of $T(\{a_i,a_{i+1},\alpha\})$. In the last instance, as $[a_i,a_{i+1}]_{T-\alpha}$ is not an interval of T, there is $x_i\in[a_i,a_{i+1}]_{T-\alpha}$ such that $\{a_i,x_i\}$ is not an interval of $T(\{a_i,x_i,\alpha\})$. Note that since $\{a_i,a_{i+1}\}$ is an interval of $T(\{a_i,a_{i+1},\alpha\})$, $\{x_i,a_{i+1}\}$ is not an interval of $T(\{x_i,a_{i+1},\alpha\})$ and let X_i be the singleton $\{x_i\}$.

The elements of $\{a_0,\dots,a_k\}\cup X_{\min}\cup(\bigcup_{i\in\{0,\dots,k-1\}}X_i)\cup X_{\max}$ are then denoted by $b_0,\dots,b_n$ where for $i\in\{0,\dots,n-1\}$, $b_i\rightarrow b_{i+1}$. By construction, $b_n\rightarrow\alpha\rightarrow b_0$ and for $i\in\{0,\dots,n-1\}$, $\{b_i,b_{i+1}\}$ is not an interval of $T(\{b_i,b_{i+1},\alpha\})$, that is to say, either $b_i\rightarrow\alpha\rightarrow b_{i+1}$ or $b_{i+1}\rightarrow\alpha\rightarrow b_i$. It ensues that n is odd and that $T(\{b_0,\dots,b_n\})\cup\{\alpha\}\simeq V_{n+2}$. $\square$

The section is completed by the next two remarks.

Remark 4. The class $\mathscr{S}_\infty$ is empty.

Proof. The tournaments T_ω, U_ω and V_ω, defined below, will be utilized.

- The tournament T_ω is defined on $\omega\times\{0,1\}$ as follows: for all $i<j\in\omega$, $(i,0)\rightarrow(j,0)$ and $(i,1)\rightarrow(j,1)$. Furthermore, for each $i\in\omega$, $\{(i+1,0),(i+2,0),\dots\}\rightarrow(i,1)\rightarrow\{(0,0),\dots,(i,0)\}$.
- The tournament U_ω is defined on $\omega\times\{0,1\}$ as follows: for every $i<j\in\omega$, $(i,0)\rightarrow(j,0)$ and $(j,1)\rightarrow(i,1)$. Moreover, for all $i\in\omega$, $\{(i+1,0),(i+2,0),\dots\}\rightarrow(i,1)\rightarrow\{(0,0),\dots,(i,0)\}$.
- The tournament V_ω is defined on $\omega\cup\{-1\}$ as follows: $V_\omega(\omega)$ is the usual total order on ω and $\{1,3,5,\dots\}\rightarrow(-1)\rightarrow\{0,2,4,\dots\}$.

It follows from Propositions 8–11 that for $X=P,T,U$ or V, $X_\omega \in \mathscr{X}_\infty^F$. In particular, it ensues that the tournaments P_ω, T_ω, U_ω and V_ω are indecomposable but it may be shown that they are not self-dual. Indeed, firstly, as for every $x\in\omega$, $N^-_{P_\omega}(x)$ is infinite and $N^+_{P_\omega}(x)$ is finite, P_ω is not self-dual. Secondly, it may be proven that T_ω does not embed $\omega^\star$. Suppose, on the contrary, that there is a sequence $(x_n)_{n\geqslant 0}$ of elements of $\omega\times\{0,1\}$ such that for all $n>m\geqslant 0$, $x_n\to x_m$. There would then be $i\in\{0,1\}$ and an infinite subsequence $(x_{u(n)})_{n\geqslant 0}$ of $(x_n)_{n\geqslant 0}$ such that $\{x_{u(n)}; n\geqslant 0\}\subseteq \omega\times\{i\}$; yet, $T(\omega\times\{i\})\simeq\omega$. In consequence, T_ω embeds ω without embedding $\omega^\star$ and, therefore, T_ω is not self-dual. Thirdly, assume that there is an isomorphism f from U_ω onto $(U_\omega)^\star$, since $\mathscr{C}(U_\omega)=\mathscr{C}[(U_\omega)^\star]=\omega\times\{1\}$, $f(\omega\times\{1\})=\omega\times\{1\}$; however, $U_\omega(\omega\times\{1\})\simeq\omega^\star$ is not self-dual. Lastly, as -1 is the unique element of $\omega\cup\{-1\}$ such that $V_\omega-(-1)$ is a total order, if f is an isomorphism from V_ω onto $(V_\omega)^\star$, then $f(-1)=-1$ and, thus, $f(\omega)=\omega$; yet, $V_\omega(\omega)=\omega$ is not self-dual. In another vein, notice, now and henceforth, that for $X=P,T,U$ or V, there is a proper subset Y of the set of the vertices of X_ω such that $X_\omega(Y)\simeq X_\omega$. Indeed, $P_\omega-0\simeq P_\omega$, $T_\omega-\{(0,0),(0,1)\}\simeq T_\omega$, $U_\omega-\{(0,0),(0,1)\}\simeq U_\omega$ and $V_\omega-\{0,1\}\simeq V_\omega$.

At present, returning to Remark 4, assume, by contradiction, that $\mathscr{S}_\infty$ admits at least one element $T=(V,A)$. Recall that, by Theorem 3, $\mathscr{S}_\infty$ is a subclass of $\mathscr{S}_\infty^F$ and, hence, by Corollary 5, $T\in(\mathscr{S}_\infty\cap\mathscr{P}_\infty^F)\cup(\mathscr{S}_\infty\cap\mathscr{T}_\infty^F)\cup(\mathscr{S}_\infty\cap\mathscr{U}_\infty^F)\cup(\mathscr{S}_\infty\cap\mathscr{V}_\infty^F)$. Consequently, it suffices to envisage three cases.

- If $T\in(\mathscr{S}_\infty\cap\mathscr{P}_\infty^F)$, then, by Proposition 8, T is isomorphic to P_ω, to $(P_\omega)^\star$ or to $P_{\omega^\star+\omega}$. In consequence, by interchanging T and $T^\star$, it may be supposed that T embeds P_ω and, by the above observation, T properly embeds P_ω; however, P_ω is indecomposable without being self-dual.
- If $T\in(\mathscr{S}_\infty\cap\mathscr{T}_\infty^F)$ (resp. $T\in(\mathscr{S}_\infty\cap\mathscr{U}_\infty^F)$), then, by Proposition 9 (resp. Proposition 10), there is an infinite subset X of V such that $T(X)$ is a total order and there is a function φ which associates with each $x\in V-X$ a nontrivial initial interval of $T(X)$ and which satisfies the three assertions enunciated in said proposition. Since any infinite total order embeds ω or $\omega^\star$, by considering T in the place of $T^\star$, it may be assumed that there is a sequence $(\alpha_n)_{n\geqslant 0}$ of elements of X such that for every $n>m\geqslant 0$, $\alpha_m\to\alpha_n$. Moreover, for all $n\geqslant 0$, there is $\beta_n\in V-X$ such that $\alpha_n\in\varphi(\beta_n)$ and $\alpha_{n+1}\notin\varphi(\beta_n)$. It is then easy to verify that the function which associates, for each $n\geqslant 0$, $(n,0)$ with α_n and $(n,1)$ with β_n, is an isomorphism from $T(\{\alpha_n;n\geqslant 0\}\cup\{\beta_n;n\geqslant 0\})$ onto T_ω (resp. U_ω). It follows that T embeds T_ω (resp. U_ω) and, by the above comment, T properly embeds T_ω (resp. U_ω); yet, T_ω (resp. U_ω) is indecomposable without being self-dual.
- If $T\in(\mathscr{S}_\infty\cap\mathscr{V}_\infty^F)$, then, by Proposition 11, there is $\alpha\in V$ such that $T-\alpha$ is a total order which fulfills both properties enunciated in this proposition. Since any infinite total order embeds ω or $\omega^\star$, by interchanging T and $T^\star$, it may be supposed that there is a sequence $(a_i)_{i\geqslant 0}$ of elements of $V-\{\alpha\}$ such that for every $i>j\geqslant 0$, $a_j\to a_i$. As in the proof of Proposition 11, from elements $a_0,\ldots,a_i,\ldots$ are defined the subsets $X_{\min},X_0,\ldots,X_i,\ldots$ of $V-\{\alpha\}$ and it is easy to verify that $T[\{\alpha\}\cup X_{\min}\cup$

$\{a_i; i \geqslant 0\} \cup (\bigcup_{i \geqslant 0} X_i)] \simeq V_\omega$. In consequence, T embeds V_ω and, by the above observation, T properly embeds V_ω; however, V_ω is indecomposable without being self-dual. $\square$

Remark 5. Corollary 4 is not valid in the infinite case. Indeed, the tournament P_ω is indecomposable without being self-dual. Yet, for all subsets X of ω, such that $6 \leqslant |X| \leqslant 10$ and $P_\omega(X)$ is indecomposable, $P_\omega(X) \simeq P_{|X|}$ and, thus, $P_\omega(X)$ is self-dual.

5. A new mode of reconstruction

In the section, unless indication to the contrary, only finite tournaments will be considered. In the first place, the classical definition of the reconstructibility of Ulam [24] is reiterated. A tournament $T = (V, A)$ is said to be *reconstructible* provided that for every tournament $T' = (V', A')$, if there exists a one-to-one correspondence f from V onto V', such that for each $x \in V$, $T - x \simeq T' - f(x)$, then $T \simeq T'$. Equivalently, a tournament $T = (V, A)$ is reconstructible provided that for all tournaments $T' = (V, A')$, if for each $x \in V$, $T - x \simeq T' - x$, then $T \simeq T'$. One of the early results concerning the reconstruction of tournaments is attributable to Harary and Palmer [10], who proved that the nonstrongly connected tournaments of cardinality $\geqslant 5$ are reconstructible. This was one of the few positive answers to the problem of reconstruction of tournaments because, afterwards, Stockmeyer [22] constructed a family of indecomposable tournaments of arbitrarily great cardinality which are not reconstructible. Through these counter-examples to the reconstruction of tournaments, other modes of reconstruction were envisaged, among them, the I-reconstruction defined in the following way. Given a set I of integers, let $T = (V, A)$ and $T' = (V, A')$ be tournaments such that for every $i \in I$, $|i| < |V|$. The tournaments T and T' are said to be *I-hypomorphic* provided that for each subset X of V, if $|X| \in I$ (resp. $-|X| \in I$), then $T(X) \simeq T'(X)$ (resp. $T - X \simeq T' - X$). A tournament is then said to be *I-reconstructible* provided that for all tournaments T', if T and T' are I-hypomorphic, then $T \simeq T'$. For example, the $\{1, \ldots, k\}$-reconstruction was introduced by Fraïssé [5] and Lopez [14] showed that the tournaments are $\{3, \ldots, 6\}$-reconstructible. Later, Pouzet [18] introduced the $\{-k\}$-reconstruction and Lopez and Rauzy [15,16] proved that the tournaments are $\{-4\}$-reconstructible. Lastly, it is pointed out that the problem of the $\{-1, -2, -3\}$-reconstruction of tournaments is still open.

The purpose of the section is to introduce a new manner of reconstruction. Indeed, instead of considering all of the subtournaments of a given cardinality, only the subtournaments of a certain type may be taken into account. More precisely, given a class $\mathscr{D}$ of tournaments, the tournaments $T = (V, A)$ and $T' = (V, A')$ are said to be *$\mathscr{D}$-hypomorphic* provided that for each proper subset X of V, if $T(X) \in \mathscr{D}$ or if $T'(X) \in \mathscr{D}$, then $T(X) \simeq T'(X)$. A tournament T is then said to be *$\mathscr{D}$-reconstructible* provided that for every tournament T', if T and T' are $\mathscr{D}$-hypomorphic, then $T \simeq T'$.

In what follows, the $\mathscr{I}$-reconstruction of tournaments, where $\mathscr{I}$ is the class of indecomposable tournaments, will be examined. It is pointed out that, within the framework of the $\mathscr{I}$-reconstruction, results opposite to those obtained in classic reconstruction are arrived at. To begin, the below theorem linking the duality, the indecomposability and the reconstruction of tournaments is recalled.

Theorem 4 (Boussairi et al. [1]). *Given tournaments $T=(V,A)$ and $T'=(V,A')$, if T and T' are $\{3\}$-hypomorphic and if T is indecomposable, then $T'=T$ or $T'=T^{\star}$.*

The principal result of the section may be stated as follows:

Theorem 5. *Every indecomposable tournament of cardinality $\geqslant 11$ is $\mathscr{I}$-reconstructible.*

Proof. Given an indecomposable tournament $T=(V,A)$ such that $|V|\geqslant 11$, let $T'=(V,A')$ be a tournament such that T and T' are $\mathscr{I}$-hypomorphic. As T and T' are $\mathscr{I}$-hypomorphic, $\{X\subseteq V\colon T(X)\simeq P_3\}=\{X\subseteq V\colon T'(X)\simeq P_3\}$ and, therefore, T and T' are $\{3\}$-hypomorphic. It then follows from Theorem 4 that $T'=T$ or $T^{\star}$. In the second instance, for all $X\subset V$, if $T(X)$ is indecomposable, then, by the definition of the $\mathscr{I}$-hypomorphy, $T(X)\simeq T'(X)$ or, equivalently, $T(X)$ is self-dual. In consequence, $T\in\mathscr{S}_{|V|}$ and, by Theorem 2, $T\in\mathscr{C}_{|V|}\cup\{P_{|V|}\}$. In particular, T is self-dual, that is to say, $T'\simeq T$. $\square$

By using Corollary 4 in the preceding proof, it may be noted that each indecomposable tournament of cardinality $\geqslant 11$ is $\mathscr{I}_{(\leqslant 10)}$-reconstructible, where $\mathscr{I}_{(\leqslant 10)}$ denotes the class of indecomposable tournaments of cardinality $\leqslant 10$.

On the other hand, the decomposable tournaments, even if they are strongly connected, are not $\mathscr{I}$-reconstructible. Indeed, the tournaments T and T', below defined on $\{-4,\dots,2n\}$, where $n\geqslant 2$, are $\mathscr{I}$-hypomorphic but they are not isomorphic.

- In T, $\{-3,-2,-1\}\to\{0,\dots,2n\}\to -4$ and $-4\to\{-3,-2,-1\}$.
- In T', $-4\to\{0,\dots,2n\}\to\{-3,-2,-1\}$ and $\{-3,-2,-1\}\to -4$.
- $T(\{0,\dots,2n\})=T'(\{0,\dots,2n\})=T_{2n+1}$ and $T(\{-3,-2,-1\})=T'(\{-3,-2,-1\})=-1\to -2\to -3$ and $-3\to -1$.

The section is completed with some remarks concerning the reconstruction of infinite tournaments. The infinite and indecomposable (resp. decomposable) tournaments are not $\mathscr{I}$-reconstructible. Indeed, P_ω and $P_\omega^{\star}$ (resp. ω and $\omega^{\star}$) are $\mathscr{I}$-hypomorphic without being isomorphic. On the other hand, the infinite and indecomposable tournaments are $(\mathscr{I}\cup\mathscr{I}_\infty)$-reconstructible where $\mathscr{I}_\infty$ is the class of the infinite and indecomposable tournaments. Indeed, given an infinite and indecomposable tournament $T=(V,A)$, let $T'=(V,A')$ be a tournament such that T and T' are $(\mathscr{I}\cup\mathscr{I}_\infty)$-hypomorphic. As T and T' are $\mathscr{I}$-hypomorphic, T and T' are $\{3\}$-hypomorphic. It ensues from Theorem 3 that Theorem 4 is still valid in the infinite case and, consequently, either $T'=T$ or $T'=T^{\star}$. However, if $T'=T^{\star}$, then T and $T^{\star}$ are $(\mathscr{I}\cup\mathscr{I}_\infty)$-hypomorphic or, equivalently, $T\in\mathscr{S}_\infty$, which contradicts Remark 4. Lastly, it is noted that the infinite and decomposable tournaments are generally not $(\mathscr{I} \cup \mathscr{I}_\infty)$-reconstructible because two total orders defined on the same set are $(\mathscr{I} \cup \mathscr{I}_\infty)$-hypomorphic without necessarily being isomorphic.

Appendix A.

As stated, the subject of the appendix is to verify that $\overline{\mathscr{S}_6} = \{S_1^6, S_2^6, S_3^6\}$ and that $\overline{\mathscr{S}_7} = \mathscr{C}_7 \cup \{S_1^7, S_2^7, S_3^7, S_4^7\}$.

A.1. Characterization of $\overline{\mathscr{S}_6}$

It may be directly verified that for $i \in \{1,2,3\}$, $S_i^6 \in \overline{\mathscr{S}_6}$. Inversely, consider an element $T = (\{0,\dots,5\}, A)$ of $\overline{\mathscr{S}_6}$. As T is indecomposable and self-dual, it follows from Proposition 1 that $s(T) = (2,2,2,3,3,3)$, $(1,2,2,3,3,4)$ or $(1,1,2,3,4,4)$. In another respect, since T is of even cardinality, T is not critical and it may be assumed, for example, that $T - 5$ is indecomposable and, therefore, $T - 5 \simeq P_5$, T_5 or U_5. As $s(T_5) = (2,2,2,2,2)$, it follows from Lemma 8 that $T - 5 \not\simeq T_5$. Furthermore, by considering $T^\star$ in the place of T, it may be supposed that $d_T(5) = 1$ or 2. Two cases are distinguished.

1. $T - 5 = P_5$. As $d_{T-5}(0) = d_{T-5}(1) = 1$, $d_T(0), d_T(1) \in \{1,2\}$ and, thus, $\{0,1,5\} \subseteq \{i \in \{0,\dots,5\}\colon d_T(i) \in \{1,2\}\}$. Moreover, since T is self-dual, $|\{i \in \{0,\dots,5\}\colon d_T(i) \in \{1,2\}\}| = |\{i \in \{0,\dots,5\}\colon d_T(i) \in \{3,4\}\}|$ and, furthermore, $T(\{i \in \{0,\dots,5\}\colon d_T(i) \in \{1,2\}\}) \simeq T(\{i \in \{0,\dots,5\}\colon d_T(i) \in \{3,4\}\})$. In consequence, $\{i \in \{0,\dots,5\}\colon d_T(i) \in \{1,2\}\} = \{0,1,5\}$, $\{i \in \{0,\dots,5\}\colon d_T(i) \in \{3,4\}\} = \{2,3,4\}$ and since $d_{T-5}(2) = 2$, $2 \to 5$. Moreover, as $T(\{2,3,4\}) \simeq P_3$, $T(\{0,1,5\}) \simeq P_3$ or, equivalently, $1 \to 5 \to 0$. Consequently, $d_T(0) = 1$, $d_T(1) = 2$ and, hence, $s(T) \neq (2,2,2,3,3,3)$. If $s(T) = (1,2,2,3,3,4)$, then $d_T(5) = 2$ and either $4 \to 5 \to 3$ or $3 \to 5 \to 4$. Since, in the first instance, $\{2,5\}$ is an interval of T, $3 \to 5 \to 4$ and $T = S_1^6$. If $s(T) = (1,1,2,3,4,4)$, then $d_T(5) = 1$ and $d_T(3) = d_T(4) = 4$. It follows that $\{3,4\} \to 5$ and $T = P_{(5,0,\dots,4)} \simeq S_2^6$.
2. $T - 5 = U_5$. As $\{i \in \{0,\dots,4\}\colon d_{T-5}(i) \geqslant 3\} = \{4\}$, $s(T) \neq (1,1,2,3,4,4)$. To begin, assume that $s(T) = (1,2,2,3,3,4)$. In this instance, since $\{i \in \{0,\dots,5\}\colon d_T(i) = 4\} \neq \emptyset$, $d_T(4) = 4$ or, equivalently, $4 \to 5$. If $d_T(5) = 1$, then for $i \in \{0,\dots,4\}$, $d_T(i) \geqslant 2$ and, since $d_{T-5}(3) = 1$, $d_T(3) = 2$, that is to say, $3 \to 5$. Next, α denotes the single element of $\{0,1,2\}$ such that $d_T(\alpha) = 2$ or, equivalently, such that $\{0,1,2\} - \{\alpha\} \to 5 \to \alpha$. It ensues that $\{i \in \{0,\dots,5\}\colon d_T(i) \in \{1,2\}\} = \{\alpha,3,5\}$ and $\{i \in \{0,\dots,5\}\colon d_T(i) \in \{3,4\}\} = \{0,1,2,4\} - \{\alpha\}$. As T is self-dual, $T(\{\alpha,3,5\}) \simeq T(\{0,1,2,4\} - \{\alpha\})$. Since for $\alpha = 0$ or 2, $T(\{\alpha,3,5\}) \not\simeq T(\{0,1,2,4\} - \{\alpha\})$, $\alpha = 1$. It follows that $T - 3 = P_{(5,1,2,4,0)}$, which allows for a return to the preceding case. At present, suppose that $d_T(5) = 2$ and, thus, that $d_T(3) = 1$ or, equivalently, $5 \to 3$. As before, α denotes the unique element of

$\{0,1,2\}$ such that $d_T(\alpha)=2$, that is to say, such that $\{0,1,2\}-\{\alpha\} \to 5 \to \alpha$. It ensues that $\{i\in\{0,\dots,5\}: d_T(i)\in\{1,2\}\}=\{\alpha,3,5\}$ and $\{i\in\{0,\dots,5\}: d_T(i)\in\{3,4\}\}=\{0,1,2,4\}-\{\alpha\}$. As T is self-dual, $T(\{\alpha,3,5\})\simeq T(\{0,1,2,4\}-\{\alpha\})$ and, as $5\to\{1,\alpha\}$, $T(\{0,1,2,4\}-\{\alpha\})\simeq O_3$ and, necessarily, $\alpha=2$. However, if $\alpha=2$, then $\{1,5\}$ is an interval of T. Finally, assume that $s(T)=(2,2,2,3,3,3)$ and, hence, that $d_T(5)=2$. As $d_{T-5}(3)=1$ (resp. $d_{T-5}(4)=3$), $d_T(3)=2$ (resp. $d_T(4)=3$) or, equivalently, $3\to 5$ (resp. $5\to 4$). As before, by denoting by α the only element of $\{0,1,2\}$ such that $d_T(\alpha)=2$, $T(\{\alpha,3,5\})\simeq T(\{0,1,2,4\}-\{\alpha\})$. Yet, if $\alpha=0$ (resp. $\alpha=2$), then $T(\{0,3,5\})\simeq O_3$ (resp. $T(\{2,3,5\})\simeq P_3$) and $T(\{1,2,4\})\simeq P_3$ (resp. $T(\{0,1,4\})\simeq O_3$). In consequence, $\alpha=1$, that is to say, $\{0,2,3\}\to 5\to\{1,4\}$ and, thus, $T=S_3^6$.

A.2. Characterization of $\overline{\mathscr{S}_7}$

The characterization of $\overline{\mathscr{S}_7}$ is begun with an observation relating to the construction of an element of $\overline{\mathscr{S}_6}$ from an element of $\overline{\mathscr{S}_5}$. Let $T=(\{0,\dots,4\}\cup\{\alpha\},A)$ be an element of $\overline{\mathscr{S}_6}$ such that $T(\{0,\dots,4\})=P_5$. By the foregoing, T is isomorphic to S_1^6, to S_2^6 or to S_3^6. Since the permutation $(0,3)(1,2)(4,5)$ is an isomorphism from S_3^6 onto $(S_3^6)^\star$, since $\mathrm{Aut}(S_3^6)$ is generated by the permutation $(1,3,5)(2,0,4)$ and since $S_3^6-5\simeq U_5$, for $i\in\{0,\dots,5\}$, $S_3^6-i\simeq U_5$. In consequence, T is isomorphic to S_1^6 or to S_2^6. If $d_T(\alpha)\in\{1,2\}$, then it ensues from the characterization of $\overline{\mathscr{S}_6}$ that either $T\simeq S_1^6$ and $\{1,2,3\}\to\alpha\to\{0,4\}$ or $T\simeq S_2^6$ and $T=P_{(\alpha,0,\dots,4)}$. On the other hand, if $d_T(\alpha)\in\{3,4\}$, then, by interchanging T and $T^\star$ while conserving $T(\{0,\dots 4\})=P_5$, it is obtained that either $T\simeq S_1^6$ and $\{0,4\}\to\alpha\to\{1,2,3\}$ or $T\simeq S_2^6$ and $T=P_{(0,\dots,4,\alpha)}$. At present, let $T=(\{0,\dots,4\}\cup\{\alpha\},A)$ be an element of $\overline{\mathscr{S}_6}$ such that $T(\{0,\dots,4\})=U_5$ and $T\simeq S_3^6$. If $d_T(\alpha)\in\{1,2\}$, then it follows from the preceding characterization that $\{0,2,3\}\to\alpha\to\{1,4\}$. On the other hand, if $d_T(\alpha)\in\{3,4\}$, then, by considering $T^\star$ in the place of T while conserving $T(\{0,\dots,4\})=U_5$, it is obtained that $\{1,3\}\to\alpha\to\{0,2,4\}$.

It follows from Lemmas 3 and 4 that $\{S_1^7\}\cup\mathscr{C}_7\subseteq\overline{\mathscr{S}_7}$ and it may be directly verified that for $i\in\{2,3,4\}$, $S_i^7\in\overline{\mathscr{S}_7}$. Inversely, let $T=(\{0,\dots,6\},A)$ be an element of $\overline{\mathscr{S}_7}-\mathscr{C}_7$. Since T is not critical, there is $x\in\{0,\dots,6\}$ such that $T-x$ is indecomposable. As $T\in\overline{\mathscr{S}_7}$, $T-x\in\overline{\mathscr{S}_6}$ and it follows from the characterization of $\overline{\mathscr{S}_6}$ that $T-x$ is isomorphic to S_1^6, to S_2^6 or to S_3^6. So, suppose that $x=6$ and that $T-x=S_1^6$, S_2^6 or S_3^6. Three cases are envisaged.

1. $T-6=P_6$. It ensues from Lemma 9 that either $T\simeq S_i^6$, where $i\in\{1,2,3\}$, or $6\in\mathrm{Ext}(X)$ where $X=\{0,\dots,4\}$. If $6\in\mathrm{Ext}(X)$, then, by the above observation, it is sufficient to consider these three subcases.
   - If $T-5=P_{(0,\dots,4,6)}$, then $\{5,6\}$ is an interval of T, which contradicts the indecomposability of T.
   - If $T-5=P_{(6,0,\dots,4)}$ and if $6\to 5$, then $T-4$ is indecomposable without being self-dual. It follows that if $T-5=P_{(6,0,\dots,4)}$, then $5\to 6$ and, thus, $T=P_{(6,0,\dots,5)}$.

- If $\{1,2,3\} \to 6 \to \{0,4\}$ or if $\{0,4\} \to 6 \to \{1,2,3\}$, then, whatever the arc between 5 and 6, T is not self-dual.

2. $T-6=S_1^6$. It will be proven that there is $x \in \{0,\dots,6\}$ such that $T-x \simeq P_6$, which allows for a return to the preceding case. To commence, three subcases are envisaged where X denotes $\{0,\dots,4\}$.
   - If $6 \in X(i)$, where $i=2$ or 3 (resp. $i=0$), then either $6 \to 5$ (resp. $5 \to 6$) and T is decomposable or $5 \to 6$ (resp. $6 \to 5$) and $T-i$ is indecomposable without being self-dual.
   - If $6 \in X(1)$, then, since T is indecomposable and since $1 \to 5$, $5 \to 6$. Yet, if $6 \to 1$, then T is not self-dual and, if $1 \to 6$, then $T-4$ is indecomposable without being self-dual.
   - If $6 \to X$, then either $6 \to 5$ and T is decomposable or $5 \to 6$ and $T-4$ is indecomposable without being self-dual.

   Consequently, $6 \notin \bigcup_{i \in \{0,\dots,3\}} X(i)$ and, if $6 \in [X]$, then $X \to 6$. It then follows from Lemma 2 that either $6 \in X(4)$ or $X \to 6$ or $6 \in \mathrm{Ext}(X)$. In the first instance, since T is indecomposable and since $5 \to 4$, $6 \to 5$ and, thus, $T-4=P_{(5,0,1,2,3,6)}$. In the second one, since T is indecomposable, $6 \to 5$ and, hence, $T-4=P_{(6,5,0,1,2,3)}$. In the third one, it follows from the above observation that either $T-5 \simeq P_6$ or $\{1,2,3\} \to 6 \to \{0,4\}$ or $\{0,4\} \to 6 \to \{1,2,3\}$. However, if $\{1,2,3\} \to 6 \to \{0,4\}$, then $\{5,6\}$ is an interval of T, which contradicts the indecomposability of T. Furthermore, if $\{0,4\} \to 6 \to \{1,2,3\}$, then, whatever the arc between 5 and 6, $T-4$ is indecomposable without being self-dual.

3. $T-6=S_3^6$. To begin, by denoting $\{0,\dots,4\}$ by X, if $6 \in X(i)$, where $i \in \{0,2,3\}$ (resp. $i=1$), then either $6 \to 5$ (resp. $5 \to 6$) and T is decomposable or $5 \to 6$ (resp. $6 \to 5$) and $T-i$ is indecomposable without being self-dual. Moreover, if $6 \to X$ (resp. $X \to 6$), then either $6 \to 5$ (resp. $5 \to 6$) and T is decomposable or $5 \to 6$ (resp. $6 \to 5$) and T is not self-dual. Therefore, it ensues from Lemma 2 that $6 \in X(4) \cup \mathrm{Ext}(X)$. Firstly, if $6 \in X(4)$, then, since T is indecomposable and since $5 \to 4$, $6 \to 4$ and, hence, $T-4 \simeq S_1^6$. Secondly, if $6 \in \mathrm{Ext}(X)$, then, by the above observation, either $T-5 \simeq S_i^6$, where $i \in \{1,2\}$, which allows for a return to the first two cases, or $\{0,2,3\} \to 6 \to \{1,4\}$ or $\{1,3\} \to 6 \to \{0,2,4\}$. Yet, if $\{0,2,3\} \to 6 \to \{1,4\}$, then $\{5,6\}$ is an interval of T, which contradicts the indecomposability of T. On the other hand, if $\{1,3\} \to 6 \to \{0,2,4\}$, then either $6 \to 5$ and $T-1 \simeq S_1^6$, which allows for a return to the second case, or $5 \to 6$ and $T=S_4^7$.

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